Internet Appendix for:
Strengthening and Streamlining Bank Capital Regulation*

Robin Greenwood
Harvard Business School and NBER

Samuel G. Hanson
Harvard Business School and NBER

Jeremy C. Stein
Harvard University and NBER

Adi Sunderam
Harvard Business School and NBER

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A. Formal Model

A.I. Model Overview

Our formal model makes the following key assumptions[1]:

- There are diminishing marginal social benefits from bank lending or, more generally, risk taking on the asset side. For simplicity, we assume that the market for bank lending is frictionless, so banks fully internalize the social benefits from lending.
- We assume that bank insolvency leads to banking crises that are costly for society. In addition, we assume that, in the absence of regulation, banks would not fully internalize the costs of their own insolvency due to the existence of fire-sale and credit-crunch externalities. We assume that the probability of future insolvency is increasing in risky lending and is decreasing in bank equity. For simplicity, we assume that the probability of future insolvency depends solely at the ratio of bank equity to a risk-weighted linear combination of bank assets—i.e., $k \equiv E \div (\sum_{i=1}^{N} w_i A_i)$ is a sufficient statistic for the probability of default.
- We assume that there is a social "stock cost" associated with *having* more bank equity capital. Our preferred interpretation of these stock costs is that requiring banks to finance themselves with more equity entails foregoing a valuable set of monetary services that agents enjoy when they hold bank deposits and other safe, short-term debt. The associated convenience premium on deposits and short-term debt means that the

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[1] Our model can be seen as an elaboration of the model sketched in Kashyap and Stein (2004).

Modigliani-Miller (1958) irrelevance result fails for society as a whole. However, for simplicity, we assume that the private stock costs associated with having more equity equal the social costs. Implicitly, these means that, for instance, we are ignoring the tax deductibility of interest which drives a wedge between the private and social cost of having more equity.

- We assume that, in the short-run, there is some social "flow cost" associated with *raising* more equity capital from outside investors. In other words, we assume that the regulator thinks that it may be social costly to force banks to rapidly recapitalize following a large loss. However, we assume that private flow costs associated with raising more equity exceed the social costs either because of asymmetric information problems between bank managers and outside investors as in Myers and Majluf (1984) or because of debt overhang problems as in Myers (1977).

Proposition 1 characterizes the first-best "steady state" of the banking industry. Since Proposition 1 applies only to the steady state, we can ignore the flow costs associated with raising external equity capital and need only consider the costs of having more equity here. The first-best steady state involves choosing the level of lending in different categories and the risk-based capital ratio. In doing so, the planner is trading off the social benefits of greater lending, the social costs of having more bank equity, and the expected costs arising from the probability of a future banking crisis.

(i). Since having equity is socially costly, the first-best steady state involves tolerating a non-zero probability of a banking crisis: the costs of driving this probability to zero in terms of foregone monetary services and foregone lending are simply too high. All else equal, the planner chooses a higher risk-based capital ratio k^* when the social costs of bank insolvency is higher, a lower risk-based capital ratio when the stock costs of having bank equity are higher, and a lower risk-based capital ratio when the social returns to bank risk-taking are higher.

(ii). Because banks don't fully internalize the social cost of a financial crisis, the unregulated private market equilibrium features too much risk-taking on the asset side and insufficient risk-based capital ratios relative to the first-best outcome.

(iii). If the risk of bank assets is perfectly observable and contractible, so there is no scope for arbitraging the rules, then a regulator can implement the first-best steady-state outcome in a decentralized way by (a) requiring banks to have the first-best ratio equity to risk-weighted assets k^*, (b) imposing the appropriate risk weights w_i on each category of risky assets, and (c) then allowing banks to choose the amount of risky assets in each category subject to a risk-based capital constraint of the form $E \geq k^* \sum_{i=1}^{N} w_i A_i$.[2]

[2] This is a somewhat special result. It only arises when (i) a risk-based capital ratio is a sufficient statistic for the likelihood of a crisis and (ii) there is no wedge between the private and social cost of having equity (e.g., due to the tax deductibility of interest). If these conditions fail, then one needs an additional regulatory tool in order to implement the first-best steady state. For instance, one can implement the first best by (a) imposing Pigouvian taxes on bank risk

Proposition 2: If there are multiple rules that determine capital charges, and a rule with risk weights other than w_i sometimes binds in equilibrium, then the resulting allocation of risk will be inefficient.

Proposition 3: In order for different banks to be bound by different ratios in equilibrium, there needs to be heterogeneity across banks. We assume that banks differ in terms of their cost efficiency at various kinds of lending and in the social cost of their default X_b.

(i). In this case, the first-best can be implemented by imposed a risk-based capital constraint of the form $E_b \geq k_b^* \times \sum_{i=1}^{N} w_i A_{bi}$ on bank *b*. The optimal risk-based ratio for bank *b* is increasing in the social cost given default X_b and is decreasing in bank *b*'s efficiency in lending. While capital charges for a given asset—i.e., $K_{bi}^* = k_b^* w_i$—differ across banks, the relative capital charge for any pair of assets is the same for all banks—i.e., comparing the relative capital charges for assets *i* and *j* between banks *a* and *b* we have $K_{bi}^* / K_{bj}^* = w_i / w_j = K_{ai}^* / K_{aj}^*$.

(ii). A non-risk-based leverage ratio that binds for some banks in equilibrium causes two types of distortions relative to the first-best:

a. Banks that are bound by a leverage ratio will overinvest in high-risk assets (i.e., assets that face lower capital charges under the leverage ratio than under the risk-based ratio) and underinvest in low-risk assets (i.e., assets that face higher capital charges under the leverage ratio).

b. Banks that are bound by a risk-based ratio will tend to overinvest in low-risk assets (because the underinvestment by leverage-bound banks pushes up the equilibrium returns to low-risk assets) and under-invest in high-risk assets (because overinvestment by leverage-bound banks pushes down the equilibrium returns to high-risk assets).

Proposition 4 is about the first-best "transition path" back to the steady state following a negative shock to the level of bank equity capital. The simplest way to think about the transition path is to imagine that we are at some $t = 0$ where the level of equity capital has fallen below the first-best because of an adverse shock. And, we assume that at $t = 2$ banks will "earn their way out" of this hole and we will arrive back at the first-best steady state. Thus, the regulator is only deciding what will happen at $t = 1$—i.e., deciding on the transition path back to steady state. As a result, the flow costs associated with raising equity are front and center in Proposition 4. The first-best transition path involves choosing the level of both lending and equity at $t = 1$. In doing so, the social planner is trading off the social benefits of greater lending at $t = 1$, the social costs of having

taking and, to earn the extra degree of freedom, by (b) providing a Pigouvian subsidy to equity. Alternately, one can implement the first-best by (a) imposing an appropriate risk-based capital standard and, to earn the extra degree of freedom, by requiring banks to hold the first-best level of equity.

more bank equity at $t = 1$ (which are applied to the level of equity at $t = 1$, E_1), the flow costs of raising more bank equity from $t = 0$ to $t = 1$ (which are applied to the size of the recapitalization, $E_1 - E_0$), and expected costs arising from the probability of an interim banking crisis because of some shock that lands right after $t = 1$.

(ii). If there are no social flow costs associated with raising equity capital (e.g., if the flow costs are only private in nature), then, following an adverse shock, the regulator should force banks to immediately recapitalize back to the first-best steady-state level. By way of analogy to *q*-theory, there needs to be some kind of social flow adjustment cost; otherwise it is optimal to just immediately go back to the first-best steady state.

(iii). If the social flow costs associated with raising equity capital are infinite, then equity is fixed at $t = 1$ (i.e., $E_1 = E_0$) and things are as in the model from Kashyap and Stein (2003). Specifically, the first-best involves trading off the benefits of greater lending at $t = 1$ and the costs from a higher probability of failure in the near term. And, since the level of equity is fixed here, this can be implemented perfectly using a risk-based capital ratio. In particular, the regulator should let the ratio of equity to risk-weighted assets decline at $t = 1$, tolerating a higher probability of failure than it would in the steady state. It is optimal to allow capital ratios to decline because the social costs in terms of foregone lending from maintaining the same probability of failure are too high.

(iv). Now assume that the social flow costs of raising equity are interior—i.e., they are neither zero nor infinite. Then the first best transition path involves:

a. Forcing banks to recapitalize somewhat at $t = 1$, but not forcing pushing them all the way back to the first-best steady-state level of equity.

b. Under natural regularity conditions on the proportional curvature of $f_i(A_i)$ and $\pi(k)$, this involves both (i) allowing the banking system to operate with a higher crisis probability than in steady state—i.e. relaxing the risk-based capital requirement and (ii) tolerating a decline in lending relative to the steady state.

c. Due to the wedge between the private and social costs of raising equity, the first-best transition path cannot be implemented using a risk-based capital ratio alone. Instead, to implement the first-best transition path the regulator needs to (i) force banks to issue the sufficient level of equity and (ii) impose an appropriate risk-based capital constraint. In our model, the required risk-based ratio falls during the transition path but the risk weights remain unchanged. In this way, Proposition 4 motivates using the stress test to promote dynamic resilience following an adverse shock. This is precisely how the stress tests were used in 2009.

A.II. Steady state analysis

Social welfare

For simplicity, we initially assume that the banking system consists of single representative bank. Social welfare is:

$$W = \sum_{i=1}^{N} f_i(A_i) - c(E) - X\pi(k)$$

- $f_i(A_i)$ represents the risk-adjusted net return to assets in category *i*. Specifically, we assume that $f_i(A_i) = F_i(A_i) - (1+r_i)A_i$ where $F_i'(A_i) > 0$, $F_i''(A_i) < 0$, and r_i is the appropriate risk-adjusted hurdle rate for loans in category *i*.
- $c(E)$ is the social cost of having bank equity capital (we require $E \geq 0$) and satisfies $c'(E) > 0$ and $c''(E) \geq 0$.
- *X* is the social cost of a banking crisis and $\pi(k)$ is the probability of a banking crisis where

  $$k \equiv \frac{E}{\sum_{i=1}^{N} w_i A_i}$$

  and w_i represents the risk-contribution of loans in category *i*. In other words, we assume that a risk-based capital ratio is a sufficient statistic for the probability of bank insolvency and failure. We assume that (i) $\pi(k)$ for $k > \bar{k}$ and (ii) that, for all $k \leq \bar{k}$, $\pi(k) > 0,\ \pi'(k) < 0,$ and $\pi''(k) > 0$.[3]

One can either think of the E and A_i variables as being in levels or as being scaled by some fixed level of assets in the banking system. In this latter case, the A_i are asset portfolio weights and E is the ratio of equity to total, non-risk-based assets, and the implicit assumption is that remaining $(1-\sum_{i=1}^{N} A_i)$ fraction of bank assets is held in riskless assets and that this generates zero risk-adjusted net returns.

Social planner's problem

We can write the social planner's problem as

$$\max_{k,\{A_i\}_{i=1}^{N}} \left\{ \sum_{i=1}^{N} f_i(A_i) - c(k\sum_{i=1}^{N} w_i A_i) - X\pi(k) \right\}.$$

The first-order conditions for the optimal amounts of risky assets in in categories $i = 1, \ldots, N$ are given by:

$$f_i'(A_i^*) = k^* w_i \times c'(\sum_{j=1}^{N} w_j A_j^*)$$

In words, the planner trades off the net risk-adjusted return of additional assets in category *i* versus the incremental cost of the optimal amount of equity capital needed to support those assets. The first order condition for the optimal risk-based capital ratio is given by

$$-X\pi'(k^*) = (\sum_{i=1}^{N} w_i A_i^*) \times c'(k^* \sum_{i=1}^{N} w_i A_i^*).$$

[3] As in Gordy (2003), this functional form can be rationalized if (i) all bank assets are exposed to a single systematic risk factor and (ii) all idiosyncratic risk in the bank's portfolio has been diversified away. Under these strict assumptions, a bank fails if and only if $E + \sum_{i=1}^{N} w_i A_i \tilde{f} < 0$ where $\tilde{f}$ is the realization of the systematic risk factor and where w_i captures asset *i*'s exposure to the systematic risk factor. Letting $G(\cdot)$ denote the cumulative distribution function of $\tilde{f}$, the probability of bank failure is then $\pi(k) \equiv G(-k)$ where $k \equiv E \div (\sum_{i=1}^{N} w_i A_i)$. Thus, we have $\pi'(k) = -G'(-k) < 0$. And, under the assumption that $G''(\cdot) > 0$ is the relevant, left-tail region, we also have $\pi''(k) = G''(-k) > 0$.

Thus, in choosing the risk-based ratio, the planner trades off the social benefits from reducing the probability of a banking crisis versus the costs of having additional bank equity.[4]

Since $-\pi'(k^*) > 0$, we have $k^* < \bar{k}$ and $\pi(k^*) > 0$. Thus, the first-best steady state involves tolerating a non-zero probability of a banking crisis: the costs of driving this probability to zero in terms of costs of having more bank equity (i.e., foregone monetary services) and foregone lending are simply too high.

Furthermore, in determining k^*, the planner is trading off the social benefits of greater lending, the social costs of having more bank equity, and the expected costs arising from the probability of a future banking crisis. Indeed, the comparative statics for the first-best steady-state solution reflect this basic intuition:

- An increase in social cost of a banking crisis *X* raises k^* and lowers A_i^* for all i.
- An increase in the marginal returns to all forms of bank lending—e.g., if the returns to all forms of lending are $\alpha f_i(A_i)$ and α rises—lowers k^* and raises A_i^* for all i.
- Under a natural regularity condition regarding the curvature of $\pi(k)$ and $f_i(A_i)$, an increase in cost of bank equity (e.g., if the cost of having equity is $\theta_E c(E)$ and θ_E rises) lowers k^* and lowers A_i^* for all i. The regulations condition are that $-\pi''(k)k / \pi'(k) > 1$ and $-f_i''(A_i)A_i / f_i'(A_i) > 1$ for all i.

Unregulated market outcome

Suppose that banks fail to internalize fraction ϕ of the social costs of a banking crisis due to the existence of fire-sale or credit-crunch externalities. Then the unregulated market outcome is a solution to

$$\max_{k,\{A_i\}_{i=1}^N} \left\{ \sum_{i=1}^N f_i(A_i) - c(k\sum_{i=1}^N w_i A_i) - (1-\phi) X \pi(k) \right\}.$$

The relevant first order conditions are (we use one asterisk to denote the social optimum and two asterisks to denote the private optimum):

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[4] One can equivalently represent the planner's problem as

$$\max_{E,\{A_i\}} \left\{ \sum_{i=1}^N f_i(A_i) - c(E) - X\pi(E \div \sum_{i=1}^N w_i A_i) \right\}.$$

The first-order conditions for the optimal amounts of risky assets in categories $i = 1, ..., N$ are given by:

$$f_i'(A_i^*) = X \times \partial\pi(E^* \div \sum_{j=1}^N w_j A_j^*) / \partial A_i .$$

In words, the planner trades off the net risk-adjusted return of additional assets in category *i* versus the incremental increase in the probability of a banking crisis. The first order condition for the optimal amount of equity capital is

$$c'(E^*) = -X \times \partial\pi(E^* \div \sum_{j=1}^N w_j A_j^*) / \partial E ,$$

—i.e., the planner set the marginal cost of having equity to the marginal benefit of equity in terms of crisis mitigation.

$$f_i'(A_i^{**}) = k^{**} w_i \times c'(k^{**} \sum_{j=1}^{N} w_j A_j^{**})$$

and

$$-(1-\phi) X \pi'(k^{**}) = (\sum_{i=1}^{N} w_i A_i^{**}) \times c'(k^* \sum_{i=1}^{N} w_i A_i^{**}).$$

Naturally, when $\phi = 0$, we have $A_i^{**} = A_i^*$ for all i and $k^{**} = k^*$. And, it is easy to show that $\partial A_i^{**} / \partial \phi > 0$ for all i and $\partial k^{**} / \partial \phi < 0$.[5] Thus, relative to the social optimum, the unregulated market outcome features excessive bank risk taking on the asset side and insufficient risk-based equity capital ratios.

Implementing the first-best steady using risk-based capital regulation

It is immediately clear that the regulator can implement the first-best steady state by imposing a risk-based capital rule of the following form

$$E \geq k^* \times \sum_{i=1}^{N} w_i A_i$$

where k^* is the first-best risk-based capital ratio and the w_i are the corresponding risk weights. Facing such risk-based capital rule (the constraint will bind at the optimum since having equity is costly), banks then choose the amount of risky lending in each category to solve

$$\max_{\{A_i\}_{i=1}^{N}} \left\{ \sum_{i=1}^{N} f_i(A_i) - c(k^* \sum_{i=1}^{N} w_i A_i) - (1-\phi) X \pi(k^*) \right\}.$$

The first order condition for lending in category *i* is

$$f_i'(A_i^{**}) = k^* w_i \times c(k^* \sum_{j=1}^{N} w_j A_j^{**}).$$

It is then clear that banks will choose $A_i^{**} = A_i^*$.[6,7]

In passing, we note the result that the first-best can be implemented in this way is quite special and we have deliberately written the model so that it holds. This is because there are $N+1$ choice variables in our model (the equity capital ratio and risky lending in N different loan categories), but a risk-based capital rule only has N degrees of freedom (there are N capital charges for each of the risky assets) and the problem facing banks only has *N* degrees of freedom.

In general, one can only implement the first-best steady state in a model of this sort using a risk-based capital rule when (i) a risk-based capital ratio is a sufficient statistic for the financial stability externality that the planner is trying to correct and (ii) there is no wedge between the private and social cost of having equity (e.g., due to the tax deductibility of interest).

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[5] We prove this below for the case of $N = 2$ lending categories. The result for general N can be proven using the formula for the inverse of a block/partitioned matrix.

[6] The regulated bank's problem is globally concave, so any solution to the first order conditions corresponds to the unique global optimum. Furthermore, it is immediate that $A_i^{**} = A_i^*$ solves this system of first order conditions. Thus, $A_i^{**} = A_i^*$ is indeed the unique global optimum to the problem facing a regulated bank.

[7] Note that properly choosing the first-best risk-based capital ratio is analogous to a form of quantity regulation in the sense of Weitzman (1974) and is informationally quite demanding. To properly set k^*, the planner needs to compute the full first-best steady state outcome.

If these conditions fail, then one needs an additional regulatory tool (an additional regulatory degree of freedom) in order to implement the first-best steady state. For instance, if these special conditions do not hold, then one can implement the first best by (a) imposing Pigouvian tax on bank risk taking and, to earn the extra degree of freedom, by (b) providing a Pigouvian subsidy to equity. Alternately, one can implement the first-best by (c) imposing an appropriate risk-based capital standard and, to earn the extra degree of freedom, by (d) requiring banks to hold the first-best level of equity.

A.III. Risk-based and non-risk-based regulation

Multiple banks

We now introduce multiple banks. There are banks $b = 1,... B$. Due to heterogeneity in their origination and monitoring expertise, we assume that different banks differ in their technological comparative advantage in providing different types of loans. Specifically, if bank b lends an amount A_{bi} in category i, it incurs an operational cost $(\eta_{bi}/2)(A_{bi})^2$. We take the $\{\eta_{bi}\}$ parameters as exogenously given. We also assume that banks differ in the social costs of their default, X_b. Thus, social welfare is given by

$$W = \sum_{i=1}^{N} f_i(\sum_{b=1}^{B} A_{bi}) - \sum_{b=1}^{B}\sum_{i=1}^{N} \frac{\eta_{bi}}{2} L_{bi}^2 - c(\sum_{b=1}^{B} E_b) - \sum_{b=1}^{B} X_b \pi(k_b)$$

where $k_b \equiv E_b \div (\sum_{i=1}^{N} w_i A_{bi})$ is bank b's risk-based capital ratio.[8]

Planner's problem

Thus, the social planner solves

$$\max_{\{A_{bi}\}_{bi},\{k_b\}_{b=1}^{B}} \left\{ \sum_{i=1}^{N} f_i(\sum_{b=1}^{B} A_{bi}) - \sum_{b=1}^{B}\sum_{i=1}^{N} \frac{\eta_{bi}}{2} L_{bi}^2 - c(\sum_{b=1}^{B} k_b \sum_{i=1}^{N} w_i A_{bi}) - \sum_{b=1}^{B} X_b \pi(k_b) \right\}.$$

The first order condition for A_{bi} for $b = 1,..., B$ and $i = 1, ..., N$ are

$$f_i'(\sum_{a-1}^{B} A_{ai}^*) = \eta_{bi} A_{bi}^* + k_b^* w_i \times c'(\sum_{a=1}^{B} k_a^* \sum_{j=1}^{N} w_j A_{aj}^*)$$

and the first order conditions for k_b for $b = 1,..., B$ are

$$-X_b \pi'(k_b^*) = (\sum_{i=1}^{N} w_i A_{bi}^*) \times c'(\sum_{a=1}^{B} k_a^* \sum_{j=1}^{N} w_j A_{aj}^*)..$$

Other than the addition of the heterogeneous cost-efficiency terms the conditions are analogous to those in the case of a single representative bank.

Implementing the first best

[8] This formulation of the social cost of equity fits most naturally with our formulation of $c(E)$ as foregone monetary services. Thus, the marginal cost of equity depends aggregate bank equity issuance $E = \sum_{b=1}^{B} E_b$. However, everything is similar if we instead assume that the social cost of equity is given by $\sum_{b=1}^{B} c(E_b)$.

Suppose that individual banks take the marginal net returns $r_i = f_i'(\sum_{b=1}^{B} A_{bi})$ and the marginal cost of equity $c_E = c'(\sum_{b=1}^{B} k_b \sum_{i=1}^{N} w_i A_{bi})$ as given—i.e., individual banks behave as price takers. Then a regulator can implement the first-best as a decentralized market equilibrium by imposing a (bank-specific) risk-based capital requirement on bank b of the form

$$E_b = k_b^* \times \sum_{i=1}^{N} w_i A_{bi} ,$$

and then allowing each bank b to choose its lending $\{A_{bi}\}_{i=1}^{N}$.[9] In that case, bank *b* solves

$$\max_{\{A_{bi}\}_{i=1}^{N}} \left\{ \sum_{i=1}^{N} r_i A_{bi} - \sum_{i=1}^{N} \frac{\eta_{bi}}{2} L_{bi}^2 - (k_b^* \sum_{i=1}^{N} w_i L_{bi}) \times c_E - (1-\phi) X_b \pi(k_b^*) \right\}.$$

Facing such a regulation, bank b then sets

$$r_i = \eta_{bi} A_{bi}^{**} + k_b^* w_i \times c_E$$

for $i = 1, ..., N$. Using the definitions of r_i and c_E we have

$$f_i'(\sum_{a=1}^{B} A_{ai}^{**}) = \eta_{bi} A_{bi}^{**} + k_b^* w_i \times c'(\sum_{a=1}^{B} k_a^* \sum_{j=1}^{N} w_i A_{aj}^{**})$$

for all b and i. Thus, the unique regulated market equilibrium is to have $A_{bi}^{**} = A_{bi}^{*}$ for all *b* and *i*.

Imposing a redundant leverage ratio

Now suppose that, for some reason, that the regulator imposes both a risk-based standard and a non-risk-based standard on banks. Suppose the risk-based capital requirement is

$$E_b \geq k_b \times \sum_{i=1}^{N} w_i A_{bi} .$$

As discussed above, this risk-based requirement can be used to implement the first best by setting $k_b = k_b^*$. The non-risk-based leverage requirement is

$$E_b \geq l_b \times \sum_{i=1}^{N} L_{bi} .$$

By assumption, this non-risk-based requirement cannot be use to implement the first best.

For simplicity, suppose that $\phi = 1$ so banks do not internalize any of the cost of their failure. Facing both of these constraints, bank b then solves

$$\max_{E_b, \{A_{bi}\}_{i=1}^{N}} \{ \sum_{i=1}^{N} r_i A_{bi} - \sum_{i=1}^{N} \frac{\eta_{bi}}{2} A_{bi}^2 - c_E \times E_b \}$$

subject to both of these regulatory capital constraints. The Lagrangian for the bank's problem is

$$\mathrm{L}_b = \sum_{i=1}^{N} r_i A_{bi} - \sum_{i=1}^{N} \frac{\eta_{bi}}{2} L_{bi}^2 - c_E \times E_b + \lambda_b^{RBC} [E_b - k_b \times \sum_{i=1}^{N} w_i A_{bi}] + \lambda_b^{LEV} [E_b - l_b \times \sum_{i=1}^{N} A_{bi}].$$

The first order conditions for lending in categories $i = 1, ..., N$ are

$$r_i = \eta_{bi} A_{bi}^{**} + k_b w_i \times \lambda_b^{RBC} + l_b \times \lambda_b^{LEV}$$

[9] As above, properly setting k_b^{**} is quite demanding here and is akin to a form of quantity regulation in the sense that regulator needs to compute the full first-best equilibrium in order to compute k_b^{**}.

and the first order condition for E_b is

$$-c_E + \lambda_b^{RBC} + \lambda_b^{LEV} = 0 .$$

Since $c_E > 0$, both constraints cannot be slack at an optimum. And, we will only have $\lambda_b^{RBC} > 0$ and $\lambda_b^{LEV} > 0$ in the knife-edge where $k_b \times \sum_{i=1}^{N} w_i A_{bi} = l_b \times \sum_{i=1}^{N} A_{bi}$. Thus, if only bank *b*'s risk-based constraint binds we have

$$r_i = \eta_{bi} A_{bi} + k_b w_i \times c_E$$

for $i = 1,..., N$, implying a loan supply curve of the form

$$A_{bi} = \frac{r_i - k_b w_i \times c_E}{\eta_{bi}}$$

By contrast, if only bank *b*'s non-risk-based constraint binds we have

$$r_i = \eta_{bi} A_{bi} + l_b \times c_E$$

for $i = 1,..., N$, implying a loan supply curve of the form

$$A_{bi} = \frac{r_i - l_b \times c_E}{\eta_{bi}} .$$

Example with two banks and two asset classes

We now consider a simple example with two banks and two asset classes.

- Suppose that there are two assets classes 1 and 2 where $w_2 > w_1$—i.e., asset 2 is the riskier asset.
- Suppose there are two banks:
  - Bank a has a cost advantage in asset class 1: $\eta_{a1} < \eta_{b1}$.
  - Bank b has a cost advantage in asset class 2: $\eta_{b1} < \eta_{a1}$.
- Assume that $k_a^* = k_b^* = k$, $l_a = l_b = l$, and that $kw_1 < l < kw_2$. Thus, as compared to the risk-based ratio, the leverage ratio imposes a higher capital charge on asset 1 and a lower charge on asset 2.
- Assume that $c(E) = c_E \times E$—i.e. the marginal cost of equity c_E is constant.
- Assume that we have downward sloping linear demand for loan category *i* of the form $r_i = \alpha_i - \beta_i (A_{ai} + A_{bi})$. If loan rates are fixed ($r_i = \alpha_i$), then *a*'s binding leverage constraint will distort *a*'s lending. But since loan demand is infinitely elastic there is no distortion for *b*'s lending in equilibrium. Thus, for *a*'s binding leverage constraint to distort *b*'s lending in equilibrium, we need $\beta_i > 0$.

Assume we are in an equilibrium where *a*'s leverage constraint binds and *b*'s risk-based constraint binds. This gives us a system of four equations in four unknowns:

$$A_{a1} = \frac{\alpha_1 - \beta_1 (A_{a1} + A_{b1}) - l \times c_E}{\eta_{a1}}$$

$$A_{a2} = \frac{\alpha_2 - \beta_2(A_{a2} + A_{b2}) - l \times c_E}{\eta_{a2}}$$
$$A_{b1} = \frac{\alpha_1 - \beta_1(A_{a1} + A_{b1}) - kw_1 \times c_E}{\eta_{b1}}$$
$$A_{b2} = \frac{\alpha_2 - \beta_2(A_{a2} + A_{b2}) - kw_2 \times c_E}{\eta_{b2}}.$$

Solving this system of equations, the regulated market equilibrium is

$$\begin{bmatrix} A_{a1}^{**} \\ A_{a2}^{**} \\ A_{b1}^{**} \\ A_{b2}^{**} \end{bmatrix} = \begin{bmatrix} \frac{1}{\eta_{a1}\eta_{b1} + \beta_1(\eta_{a1} + \eta_{b1})}(\eta_{b1}(\alpha_1 - lc_E) + \beta_1 c_E(kw_1 - l)) \\ \frac{1}{\eta_{a2}\eta_{b2} + \beta_2(\eta_{a2} + \eta_{b2})}(\eta_{b2}(\alpha_2 - lc_E) + \beta_2 c_E(kw_2 - l)) \\ \frac{1}{\eta_{a1}\eta_{b1} + \beta_1(\eta_{a1} + \eta_{b1})}(\eta_{a1}(\alpha_1 - kw_1 c_E) + \beta_1 c_E(kw_1 - l)) \\ \frac{1}{\eta_{a2}\eta_{b2} + \beta_2(\eta_{a2} + \eta_{b2})}(\eta_{a2}(\alpha_2 - kw_2 c_E) + \beta_2 c_E(kw_2 - l)) \end{bmatrix}.$$

Contrast this with the first-best solution where the regulator only imposes a risk-based constraint on both banks. In that case, equilibrium is

$$\begin{bmatrix} A_{a1}^{*} \\ A_{a2}^{*} \\ A_{b1}^{*} \\ A_{b2}^{*} \end{bmatrix} = \begin{bmatrix} \frac{1}{\eta_{a1}\eta_{b1} + \beta_1(\eta_{a1} + \eta_{b1})}(\eta_{b1}(\alpha_1 - kw_1 c_E) \\ \frac{1}{\eta_{a2}\eta_{b2} + \beta_2(\eta_{a2} + \eta_{b2})}(\eta_{b2}(\alpha_2 - kw_2 c_E) \\ \frac{1}{\eta_{a1}\eta_{b1} + \beta_1(\eta_{a1} + \eta_{b1})}(\eta_{a1}(\alpha_1 - kw_1 c_E) \\ \frac{1}{\eta_{a2}\eta_{b2} + \beta_2(\eta_{a2} + \eta_{b2})}(\eta_{b2}(\alpha_2 - kw_2 c_E) \end{bmatrix}.$$

Comparing these two outcomes we see that

$$\begin{bmatrix} A_{a1}^{**} \\ A_{a2}^{**} \\ A_{b1}^{**} \\ A_{b2}^{**} \end{bmatrix} - \begin{bmatrix} A_{a1}^{*} \\ A_{a2}^{*} \\ A_{b1}^{*} \\ A_{b2}^{*} \end{bmatrix} = \begin{bmatrix} \frac{(\eta_{b1} + \beta_1)c_E}{\eta_{a1}\eta_{b1} + \beta_1(\eta_{a1} + \eta_{b1})}(kw_1 - l) \\ \frac{(\eta_{b2} + \beta_2)c_E}{\eta_{a2}\eta_{b2} + \beta_2(\eta_{a2} + \eta_{b2})}(kw_2 - l) \\ -\frac{\beta_1 c_E}{\eta_{a1}\eta_{b1} + \beta_1(\eta_{a1} + \eta_{b1})}(kw_1 - l) \\ -\frac{\beta_2 c_E}{\eta_{a2}\eta_{b2} + \beta_2(\eta_{a2} + \eta_{b2})}(kw_2 - l) \end{bmatrix} = \begin{bmatrix} <0 \\ >0 \\ >0 \\ <0 \end{bmatrix}.$$

Thus, assuming that $kw_1 < l < kw_2$, bank *a*'s binding leverage ratio leads to a decline in A_{a1}, an increase in A_{a2}, a rise in A_{a1}, and a decline in A_{a2}. Finally, turning to equilibrium loan rates and aggregate quantities, we have

$$r_1^{**} - r_1^* = \beta_1(L_{a1}^* + L_{b1}^* - L_{a1}^{**} - L_{b1}^{**}) = \frac{\beta_1\eta_{b1}}{\eta_{a1}\eta_{b1} + \beta_1(\eta_{a1} + \eta_{b1})} c_E(l - kw_1) > 0$$

and

$$r_2^{**} - r_2^* = \beta_2(L_{a2}^* + L_{b2}^* - L_{a2}^{**} - L_{b2}^{**}) = \frac{\beta_2\eta_{b2}}{\eta_{a2}\eta_{b2} + \beta_2(\eta_{a2} + \eta_{b2})} c_E(l - kw_2) < 0 .$$

In other words, bank *a*'s binding non-risk-based constraint leads to a decline in low-risk lending and an increase in high-risk lending. As a result, the equilibrium returns to low-risk activities will rise and those on high-risk activities will decline.

As this example shows, one needs c_E to be large relative to the ηs in order for bank *a*'s binding leverage ratio to generate large distortions in the allocation of credit provision across banks. And, c_E must be large for these distortions to have a large impact on equilibrium loan returns (interest rates).

A.IV. Dynamic resilience following an adverse capital shock

The simplest way to think about the transition path is to imagine that we are at some $t = 0$ where the level of equity capital has fallen below the first-best steady-state level E^* because of an adverse shock. And, we assume that at $t = 2$ banks will "earn their way out" and we will arrive back at the first-best steady state. Thus, the regulator is really only deciding what will happen at $t = 1$—i.e., deciding on the transition path back to steady state.

Social welfare

For simplicity, we again assume that there is just a single representative bank. We assume that welfare at $t = 1$ is

$$W_1 = \sum_{i=1}^{N} f_i(A_{i1}) - c(E_1) - \lambda(E_1 - E_0) - X\pi(k_1)$$

where we define

$$k_1 \equiv \frac{E_1}{\sum_{i=1}^{N} w_i A_{i1}} .$$

Relative to the steady-state problem described in Appendix B.II, the transition path problem adds the social flow costs of raising new external equity. These are given by $\lambda(E_1 - E_0)$ where $\lambda'(\cdot) > 0$ and $\lambda''(\cdot) \geq 0$.

Social planner's problem

The first-best transition path involves choosing the both level of lending and the level of equity capital at $t = 1$. In making this choice, the planner is trading off the social benefits of greater lending at $t = 1$, the social costs of having more bank equity at $t = 1$ (which are applied to the level

of equity at $t=1$, E_1), the social flow costs of raising more bank equity from $t=0$ to $t=1$ (which are applied to the size of the recapitalization, $E_1 - E_0$), and the expected costs arising from the probability of an interim banking crisis because of some shock that lands right after $t=1$ (this is decreasing in k_1).

Thus, the planner solves

$$\max_{\{A_{i1}\}_{i=1}^N, k_1} \left\{ \sum_{i=1}^N f_i(A_{i1}) - c(k_1 \sum_{i=1}^N w_i A_{i1}) - \lambda(k_1 \sum_{i=1}^N w_i A_{i1} - E_0) - X\pi(k_1) \right\}.$$

The first order conditions characterizing the first-best transition path are

$$f_i'(A_{i1}^*) = k_1^* w_i \times [c'(k_1^* \sum_{j=1}^N w_j A_{j1}^*) + \lambda'(k_1^* \sum_{j=1}^N w_j A_{j1}^* - E_0)]$$

for $i = 1, ..., N$ and

$$-X\pi'(k_1^*) = (\sum_{i=1}^N w_i A_{i1}^*) \times [c'(k_1^* \sum_{i=1}^N w_i A_{i1}^*) + \lambda'(k_1^* \sum_{i=1}^N w_i A_{i1}^* - E_0)].$$

Note that if $\lambda'(\cdot) = 0$, then we have $k_1^* = k^*$ and $A_{i1}^* = A_i^*$. Intuitively, if there are no social flow costs associated with raising equity capital (e.g., if the flow costs are only private in nature), then, following a shock, the regulator should force banks to immediately recapitalize back to the first-best steady-state level. By way of analogy to the neoclassical q-theory of investment, there needs to be some kind of social flow adjustment cost. If there is not, then it is optimal to just immediately go back to the first-best steady state.

If the marginal social flow costs associated with raising any positive amount of equity capital are infinite, then equity is fixed at $t=1$—i.e., $E_1^* = E_0$—and things are as in the model of Kashyap and Stein (2003). Specifically, the first-best involves trading off the benefits of greater lending at t = 1 and the costs from a higher probability of failure in the near term. Specifically, in this case where recapitalization is infeasible, we have

$$f_i'(A_{i1}^*) = -k_1^* w_i \times \frac{X\pi'(k_1^*)}{\sum_{i=1}^N w_i A_{i1}^*} = X \times \partial\pi(E_0 \div \sum_{j=1}^N w_j A_{j1}^*) / \partial A_{i1}$$

for $i = 1, ..., N$. Since the level of equity is fixed here, this can be implemented perfectly using a risk-based capital rule. In particular, the regulator should allow the ratio of equity to risk-weighted assets decline at $t=1$, tolerating a higher probability of failure than it would in the steady state. Allowing the risk-based ratio to decline is optimal because the social costs in terms of foregone lending from maintaining the same probability of failure are too high.

Now assume that the social flow costs of raising equity are interior—i.e., they are neither zero nor infinite. Then the first best transition path involves:

- Forcing banks to recapitalize somewhat at $t=1$, but not forcing pushing them all the way back to the first-best steady state level of equity.
- Under natural regularity conditions on the (proportional) curvature of $f_i(A_i)$ and $\pi(k)$, this involves both (i) allowing the banking system to operate with a higher crisis probability than in steady state—i.e. relaxing the risk-based capital requirement at $t=1$ and (ii)

tolerating a decline in lending at $t=1$ relative to the steady state. Specifically, letting $\gamma_{L_i} = -f_i''(L_i)L_i / f_i'(L_i) > 0$ and $\gamma_\pi = -\pi''(k)k/\pi'(k) > 0$. First-best lending along the transition path falls (rises) if $\gamma_\pi > 1$ ($\gamma_\pi < 1$). The first-best risk-based capital ratio along the transition path falls (rises) if $\gamma_{L_i} > 1$ ($\gamma_{L_i} < 1$) for all i.

Unregulated market outcome

Suppose that banks fail to internalize fraction ϕ of the costs of a crisis. In addition, assume that the private costs of raising equity are $(1+\theta)\lambda(E_1 - E_0)$. Thus, θ parameterizes the wedge between the private and social costs of raising additional outside equity. Under these assumptions, the unregulated market outcome is a solution to

$$\max_{k_1,\{A_{i1}\}_{i=1}^N} \left\{ \sum_{i=1}^N f_i(A_{i1}) - c(k_1 \sum_{i=1}^N w_i A_{i1}) - (1+\theta)\lambda(k_1 \sum_{i=1}^N w_i A_{i1} - E_0) - (1-\phi) X \pi(k_1) \right\}.$$

The relevant first order conditions are

$$f_i'(A_{i1}^{**}) = k_1^{**} w_i \times [c'(k_1^{**} \sum_{j=1}^N w_j A_{j1}^{**}) + (1+\theta)\lambda'(k_1^{**} \sum_{j=1}^N w_j A_{j1}^{**} - E_0)],$$

and

$$-(1-\phi) X \pi'(k_1^{**}) = (\sum_{i=1}^N w_i A_{i1}^{**}) \times [c'(k_1^{**} \sum_{i=1}^N w_i A_{i1}^{**}) + (1+\theta)\lambda'(k_1^{*} \sum_{i=1}^N w_i A_{i1}^{**} - E_0)].$$

First, we compare this to the unregulated steady state. Assuming that $\theta > 0$, then, under our regularity conditions on curvature, we will have $A_{i1}^{**} < A_i^{**}$ and $k_1^{**} < k^{**}$. Next, we compare this to the first-best transition path chosen by the social planner. The two wedges—ϕ and θ—both lead k_1^{**} to fall below k_1^{*}. However, these two wedges have opposing effects on A_{i1}^{**}. Thus, to have $A_{i1}^{**} < A_{i1}^{*}$, we need θ to be large relative to ϕ or we need $E_0 << E^{**}$ and $\lambda''(\cdot) > 0$.

Implementing the first-best transition path

Due to the wedge between the private and social costs of raising equity ($\theta > 0$), the first-best transition path cannot be implemented using a risk-based capital standard alone. The basic problem is that the private cost of recapitalization exceeds the social cost.

To see this, suppose that the regulation simply imposes a risk-based capital constraint of the form

$$E_1 = k_1^{*} \sum_{i=1}^N w_i A_{i1},$$

and allows banks to choose the lending subject to that constraint. Facing this constraint, banks then solve

$$\max_{\{A_{i1}\}_{i=1}^N} \left\{ \sum_{i=1}^N f_i(A_{i1}) - c(k_1^{*} \sum_{i=1}^N w_i A_{i1}) - (1+\theta)\lambda(k_1^{*} \sum_{i=1}^N w_i A_{i1} - E_0) - (1-\phi) X \pi(k_1^{*}) \right\}.$$

The relevant first order conditions are

$$f_i'(A_{i1}^{**}) = k_1^{*} w_i \times [c'(k_1^{*} \sum_{j=1}^N w_j A_{j1}^{**}) + (1+\theta)\lambda'(k_1^{*} \sum_{j=1}^N w_j A_{j1}^{**} - E_0)]$$

for $i = 1,..., N$. When $\theta > 0$, it is clear that banks will choose $A_{i1}^{**} < A_{j1}^{*}$. In other words, because the private costs of raising equity exceed the social costs, the severity of the credit crunch will exceed that under the first-best transition path. Thus, following an adverse shock, imposing a risk-based ratio is insufficient to implement the first-best.

Instead, to implement the first-best transition path the regulator needs to (i) force banks to issue the correct *dollar amount* of equity and (ii) impose an appropriate risk-based capital constraint. Specifically, the regulation needs to (i) force banks to issue $E_1^{**} - E_0$ dollars of equity (where $E_1^{*} = k_1^{*} \sum_{i=1}^{N} w_i A_{i1}^{*}$) and (ii) impose an risk-based capital constraint

$$E_1^{*} = k_1^{*} \times \sum_{i=1}^{N} w_i A_{i1} .$$

Facing these two constraints banks then solve

$$\max_{\{A_{i1}\}_{i=1}^{N}} \left\{ \sum_{i=1}^{N} f_i(A_{i1}) - c(E_1^{*}) - (1+\theta)\lambda(E_1^{*} - E_0) - (1-\phi) X \pi(k_1^{*}) \right\}$$

subject to this constraint. The Lagrangian for the bank's problem is

$$\mathrm{L}_1 = \sum_{i=1}^{N} f_i(A_{i1}) - c(E_1^{*}) - (1+\theta)\lambda(E_1^{*} - E_0) - (1-\phi) X \pi(k_1^{*}) + \lambda_1 [E_1^{*} - k_1^{*} \times \sum_{i=1}^{N} w_i A_{i1}].$$ The first order conditions for lending in categories $i = 1, ..., N$ are

$$f_i'(L_{i1}^{**}) = k_1^{*} \times w_i \times \lambda_1^{**} .$$

Comparing this to the first-best, it is clear that we have

$$\lambda_1^{**} = c'(E_1^{*}) + \lambda'(E_1^{*} - E_0) = c'(k_1^{*} \sum_{i=1}^{N} w_i A_{i1}^{*}) + \lambda'(k_1^{*} \sum_{i=1}^{N} w_i A_{i1}^{*} - E_0)$$

and $A_{i1}^{**} = A_{i1}^{*}$ for $i = 1, ..., N$.

A.V. Comparative statics calculations

In this section, we compute the comparative statics for our problem for the case of $N = 2$ assets. These general calculations are then sufficient to establish all of the comparative statics claims made in the prior sections. The comparative statics on the case of general N can be proved in a similar fashion using the formula for the inverse of a partitioned matrix. Letting $g(k) = -X\pi(k)$, we write the problem for $N = 2$ assets as

$$\max_{A_1, A_2, k} \{\alpha[f_1(A_1) + f_2(A_2)] + \nu g(k) - \psi c(k(w_1 A_1 + w_2 A_2))\},$$

where $f_i(\cdot) > 0$ and $f_i''(\cdot) < 0$ for $i = 1, 2$, $g'(\cdot) > 0$ and $g''(\cdot) < 0$, and $c'(\cdot) > 0$ and $c''(\cdot) \geq 0$. We will compute comparative statics for α which controls the social benefits of lending, ν which controls the social benefits of having more bank capital, and ψ which controls the social cost of having bank capital.

The three first order conditions for an optimum are

$$0 = \alpha f_1'(A_1^{*}) - \psi k^{*} w_1 c'(k^{*}(w_1 A_1^{*} + w_2 A_2^{*}))$$

$$0 = \alpha f_2'(L_2^{*}) - \psi k^{*} w_2 c'(k^{*}(w_1 A_1^{*} + w_2 A_2^{*}))$$

$$0 = \nu g'(k^*) - \psi(w_1 A_1^* + w_2 A_2^*)c'(k^*(w_1 A_1^* + w_2 A_2^*)).$$

The Hessian for this problem is

$$\mathbf{H} = \begin{bmatrix} \alpha f_1'' - \psi(kw_1)^2 c'' & -\psi(kw_1)(kw_2)c'' & -\psi w_1 c' - \psi(kw_1)(w_1 A_1 + w_2 A_2)c'' \\ -\psi(kw_1)(kw_2)c'' & \alpha f_2'' - \psi(kw_2)^2 c'' & -\psi w_2 c' - \psi(kw_2)(w_1 A_1 + w_2 A_2)c'' \\ -\psi w_1 c' - \psi(kw_1)(w_1 A_1 + w_2 A_2)c'' & -\psi w_2 c' - \psi(kw_2)(w_1 A_1 + w_2 A_2)c'' & \nu g'' - \psi(w_1 A_1 + w_2 A_2)^2 c'' \end{bmatrix}.$$

The Hessian is everywhere negative definite, so $\det(\mathbf{H}) < 0$.

By the implicit function theorem, the comparative statics for ψ are given by

$$\begin{bmatrix} \partial L_1^* / \partial \psi \\ \partial L_2^* / \partial \psi \\ \partial k^* / \partial \psi \end{bmatrix} = \mathbf{H}^{-1} \begin{bmatrix} k^* w_1 c' \\ k^* w_2 c' \\ (w_1 A_1^* + w_2 A_2^*)c' \end{bmatrix}$$

$$= \frac{c'}{\det(\mathbf{H})} \begin{bmatrix} w_1 f_2'' \times \left(\nu g'' k^* + (w_1 A_1^* + w_2 A_2^*)\psi c' \right) \\ w_2 f_1'' \times \left(\nu g'' k^* + (w_1 A_1^* + w_2 A_2^*)\psi c' \right) \\ c' k^* \psi f_2'' w_1^2 + \alpha f_1'' f_2'' A_1^* w_1 + c' k^* \psi f_1'' w_2^2 + \alpha f_1'' f_2'' A_2^* w_2 \end{bmatrix}$$

$$= \frac{c'}{\det(\mathbf{H})} \begin{bmatrix} w_1 f_2'' \nu \left(g'' k^* + g' \right) \\ w_2 f_1'' \nu \left(g'' k^* + g' \right) \\ f_2'' w_1 \alpha \left(f_1' + f_1'' A_1^* \right) + f_1'' w_2 \alpha \left(f_2' + f_2'' A_2^* \right) \end{bmatrix}$$

$$= \frac{c'}{\det(\mathbf{H})} \begin{bmatrix} -w_1 f_2'' \nu \left(-\dfrac{g'' k^*}{g'} - 1 \right) g' \\ -w_2 f_1'' \nu \left(-\dfrac{g'' k^*}{g'} - 1 \right) g' \\ -f_2'' w_1 \alpha \left(\dfrac{f_1'' A_1^*}{f_1'} - 1 \right) f_1' - f_1'' w_2 \alpha \left(\dfrac{f_2'' A_2^*}{f_2'} - 1 \right) f_2' \end{bmatrix}$$

where the third equality follows by using the first order conditions. Recall that $\det(\mathbf{H}) < 0$. Thus, if $-\pi''(k^*)k^* / \pi'(k^*) = -g''(k^*)k^* / g'(k^*) > 1$, we have $\partial A_1^* / \partial \psi < 0$ and $\partial A_2^* / \partial \psi < 0$. Similarly, if $-f_1''(A_1^*)A_1^* / f_1'(A_1^*) > 1$ and $-f_2''(A_2^*)A_2^* / f_2'(A_2^*) > 1$, we have $\partial k^* / \partial \psi < 0$. These conditions on the proportional curvature of $f_i(\cdot)$ and $g(\cdot)$ arise because k and risk-weighted assets $(w_1 A_1 + w_2 A_2)$ enter the $c(\cdot)$ function in a multiplicative fashion.

Next we compute comparative statics for ν. We have

$$\begin{bmatrix} \partial L_1^* / \partial \nu \\ \partial L_2^* / \partial \nu \\ \partial k^* / \partial \nu \end{bmatrix} = -\mathbf{H}^{-1} \begin{bmatrix} 0 \\ 0 \\ g' \end{bmatrix}$$

$$= \frac{g'}{\det(\mathbf{H})} \begin{bmatrix} \psi w_1 \alpha f_2''(c' + c'' k^* (w_1 A_1^* + w_2 A_2^*)) \\ \psi w_2 \alpha f_1''(c' + c'' k^* (w_1 A_1^* + w_2 A_2^*)) \\ -(c'' \psi \alpha f_2'' k^{*2} w_1^2 + c'' \psi \alpha f_1'' k^{*2} w_2^2 - \alpha^2 f_1'' f_2'') \end{bmatrix} = \begin{bmatrix} < 0 \\ < 0 \\ > 0 \end{bmatrix}.$$

Finally, we compute comparative statics for α. We have

$$\begin{bmatrix} \partial L_1^* / \partial \alpha \\ \partial L_2^* / \partial \alpha \\ \partial k^* / \partial \alpha \end{bmatrix} = -\mathbf{H}^{-1} \begin{bmatrix} f_1' \\ f_2' \\ 0 \end{bmatrix}$$

$$= -\frac{\psi k^*}{\alpha} c' \mathbf{H}^{-1} \begin{bmatrix} w_1 \\ w_2 \\ 0 \end{bmatrix}$$

$$= \frac{\psi k^*}{\det(\mathbf{H})} c' \begin{bmatrix} w_1 f_2''(c'' \theta (w_1 A_1^* + w_2 A_2^*)^2 - \nu g'') \\ w_2 f_1''(c'' \theta (w_1 A_1^* + w_2 A_2^*)^2 - \nu g'') \\ -\psi (f_2'' w_1^2 + f_1'' w_2^2)(c' + c'' k^* (w_1 A_1^* + w_2 A_2^*)) \end{bmatrix} = \begin{bmatrix} > 0 \\ > 0 \\ < 0 \end{bmatrix}$$

where the second equality follows from the first order conditions.