I sketch here the procedure in section 3.2 for estimating the effective required return \( r(t) \) on capital for the US corporate sector. I specify the model in continuous time, and use superscripts to denote the time \( t \) for economy of notation. I am also more explicit here about how the underlying continuous time flows are aggregated into the measured flow for a given time period.

**Assumed stochastic processes**

I assume two stochastic processes beyond what is already visible in the data:

- \( \pi_t \), which is a stationary, ergodic process for the share of gross output that goes to profits.
- \( \zeta_t \), which reflects stochastic pricing error for the total market value of corporations, with mean 1 (where 1 corresponds to no error).

**Core relations**

**First relation: profit share of flows.** We know that all non-profit income will be allocated between depreciation, labor, and the various types of capital. This is a flow relation

\[
(1 - \pi_t) Y_t = w^t L^t + \sum_i (\delta_i + r^t - g_{P_i}) P_i^t K_i^t
\]

which can be rewritten as

\[
\pi_t = 1 - \frac{w^t L^t}{Y^t} - \sum_i \frac{\delta_i P_i^t K_i^t}{Y^t} - \sum_i (r^t - g_{P_i}) \frac{P_i^t K_i^t}{Y^t}
\]  

(22)

or, if we don’t want to divide by \( Y^t \), as

\[
\pi^t Y^t = Y^t - w^t L^t - \sum_i \delta_i P_i^t K_i^t - \sum_i (r^t - g_{P_i}) P_i^t K_i^t
\]

**Consolidating into an accumulated flow.** Suppose that we write

\[
\int_t^{t+\Delta t} \pi^s Y^s \, ds = \int_t^{t+\Delta t} (Y^s - w^s L^s - \sum_i \delta_i P_i^s K_i^s) \, ds - \sum_i \int_t^{t+\Delta t} (r^s - g_{P_i}) P_i^s K_i^s \, ds
\]  

(23)
We can identify the first part as simply real net capital income during the period, while for the second term we must write
\[
\int_t^{t+\Delta t} (r^s - g_{P_s}) P_t^s K_t^s \, ds \approx \left( (r^t+\Delta t - g_{P_t}) P_t^{t+\Delta t} K_t^{t+\Delta t} + (r^t - g_{P_t}) P_t^t K_t^t \right) \frac{\Delta t}{2} \tag{24}
\]

**Second relation: asset pricing.** The expected discounted value of the profit stream from time \( t \) onward is (in real terms)
\[
Y^t \cdot \int_0^\infty e^{(g_Y - \delta_\pi)s - \int_t^{t+s} r^u du} \mathbb{E}_t[\pi^{t+s}] \, ds + \sum_i P_t^i K_t^i
\]
where \( \delta_\pi \) is the rate at which pure profits decay. (We can think of it as the rate at which a given company, for instance, on average loses the ability to make pure profits. There is no clear basis for picking \( \delta_\pi \), and I will choose \( \delta_\pi = .015 \), which implies a half-life of just below 50 years—within reason given the typical lifetimes of American corporations. Fortunately, the precise choice of \( \delta_\pi \) does not matter much for the results.)

This expected discounted value plus the value of capital itself is (again in real terms)
\[
Y^t \cdot \int_0^\infty e^{(g_Y - \delta_\pi)s - \int_t^{t+s} r^u du} \mathbb{E}_t[\pi^{t+s}] \, ds + \sum_i P_t^i K_t^i
\]
I assume that the market value of the corporate sector equals this overall value times \( \zeta_t \), the multiplicative stochastic pricing error that has mean 1, follows a stationary, ergodic process, and is drawn independently of \( \pi, \{P_t^i\}, \{K_t^i\}, \) and \( Y^t \):
\[
MV^t = \zeta_t \left( Y^t \cdot \int_0^\infty e^{(g_Y - \delta_\pi)s - \int_t^{t+s} r^u du} \mathbb{E}_t[\pi^{t+s}] \, ds + \sum_i P_t^i K_t^i \right) \tag{25}
\]
Define \( OMV^t \equiv MV^t - \sum_i P_t^i K_t^i \), and rewrite (25) as
\[
OMV^t = \zeta_t \left( Y^t \cdot \int_0^\infty e^{(g_Y - \delta_\pi)s - \int_t^{t+s} r^u du} \mathbb{E}_t[\pi^{t+s}] \, ds \right) + (\zeta_t - 1) \sum_i P_t^i K_t^i \tag{26}
\]

**Second relation, part two: taking first differences.** Now use (26) to compute
\[
\frac{\phi(t)}{Y_t^{t-1}} \left( OMV^t - e^{-\delta_\pi \Delta t - \int_t^{t+\Delta t} r^u du} \cdot OMV^{t+\Delta t} \right) \tag{27}
\]
for some small \( \Delta t \), dividing by \( Y_t^{t-1} \) (to be defined later, but known at time \( t \)) and multiplying by any deterministic function \( \phi(t) \) of \( t \). Expanding the term inside the parentheses
in (27), we obtain

\[ OMV^t - e^{-\delta_t \Delta t - \int_t^{t+\Delta t} \mu^u du} OMV^{t+\Delta t} \]
\[ = \zeta^t \left( Y^t \int_0^\infty e^{(s_Y - \delta_t) s - \int_t^{t+\Delta t} \mu^u du} \mathbb{E}_t[\pi^{t+s}] ds \right) - \zeta^{t+\Delta t} \left( e^{(s_Y - \delta_t) s - \int_t^{t+\Delta t} \mu^u du} \mathbb{E}_{t+\Delta t}[\pi^{t+s}] ds \right) \]
\[ + (\zeta^t - 1) Y^t \sum_i P_i^t K_i^t + e^{-\delta_t \Delta t - \int_t^{t+\Delta t} \mu^u du} (\zeta^{t+\Delta t} - 1) Y^{t+\Delta t} \sum_i P_{i+\Delta t} K_i^{t+\Delta t} \] (28)

Suppose now that we take the unconditional expectation of (27). Given the assumed independence of \( \zeta^t, Y^t, \) and \( \pi^t, \) (28) simplifies dramatically and we are left with

\[ \mathbb{E} \left[ \frac{\phi(t)}{Y_{t-1,t}} \left( OMV^t - e^{-\delta_t \Delta t - \int_t^{t+\Delta t} \mu^u du} OMV^{t+\Delta t} \right) \right] = \mathbb{E} \left[ \frac{\phi(t)}{Y_{t-1,t}} Y^t \int_0^{\Delta t} e^{(s_Y - \delta_t) s - \int_t^{t+\Delta t} \mu^u du} \pi^{t+s} ds \right] \] (29)

We can further manipulate (29), using the law of iterated expectations to obtain

\[ \mathbb{E} \left[ \frac{\phi(t)}{Y_{t-1,t}} Y^t \int_0^{\Delta t} e^{(s_Y - \delta_t) s - \int_t^{t+\Delta t} \mu^u du} \pi^{t+s} ds \right] = \mathbb{E} \left[ \frac{\phi(t)}{Y_{t-1,t}} \mathbb{E}_t \left[ \int_0^{\Delta t} e^{(s_Y s) Y^t} e^{-\delta_t \Delta s - \int_t^{t+s} \mu^u du} \pi^{t+s} ds \right] \right] \]
\[ = \mathbb{E} \left[ \frac{\phi(t)}{Y_{t-1,t}} \mathbb{E}_t \left[ \int_0^{\Delta t} e^{-\delta_t \Delta s - \int_t^{t+s} \mu^u du} \pi^{t+s} Y^{t+s} ds \right] \right] \]
\[ = \mathbb{E} \left[ \frac{\phi(t)}{Y_{t-1,t}} \int_0^{\Delta t} e^{-\delta_t \Delta s - \int_t^{t+s} \mu^u du} \pi^{t+s} Y^{t+s} ds \right] \] (30)

Assuming that \( \Delta t \) is small enough and \( \pi^{t+s} Y^{t+s} \) is sufficiently close to being continuous, we can approximate the integral inside (30) by

\[ \int_0^{\Delta t} e^{-\delta_t \Delta s - \int_t^{t+s} \mu^u du} \pi^{t+s} Y^{t+s} ds \approx \frac{1 + e^{-\delta_t \Delta t - (r^t + r^{t+\Delta t})/2}}{2} \int_0^{\Delta t} \pi^{t+s} Y^{t+s} ds \] (31)

**Full estimation strategy.** We have shown that the unconditional expectation of (27), which we can approximate by

\[ \mathbb{E} \left[ \frac{\phi(t)}{Y_{t-1,t}} \left( OMV^t - e^{-\delta_t \Delta t - (r^t + r^{t+\Delta t})/2} OMV^{t+\Delta t} \right) \right] \] (32)

has unconditional expectation approximately equal to

\[ \mathbb{E} \left[ \frac{\phi(t)}{Y_{t-1,t}} \cdot \frac{1 + e^{-\delta_t \Delta t - (r^t + r^{t+\Delta t})/2}}{2} \cdot \int_0^{\Delta t} \pi^{t+s} Y^{t+s} ds \right] \] (33)
where according to (23) and (24), we can obtain the approximate flow of pure profits \( \int_0^{\Delta t} \pi^{t+s} Y^{t+s} ds \) in (33) as

\[
\int_0^{\Delta t} \pi^{t+s} Y^{t+s} ds \\
\approx \int_t^{t+\Delta t} \left( Y^s - w^s L^s - \sum_i \delta_i P^s_i K^s_i \right) ds - \sum_i \left( (r^{t+\Delta t} - g) P_i^{t+\Delta t} K_i^{t+\Delta t} + (r^t - g) P_i^t K_i^t \right) \frac{\Delta t}{2}
\]

(34)

where the first term is just the recorded net return on capital in the period \([t, t + \Delta t]\) as measured in the national accounts, while the second term can be derived from the nominal quantities \(P_i K_i\) of each type of capital.

The moments (32) and (33) are equal, and we can set the corresponding sample moments equal to each other. Generally I will look at an annual frequency, such that \(\Delta t = 1\). Given a functional form for \(r^t\) with \(n\) free parameters to be pinned down, we can choose \(n\) functions for \(\phi(t)\) to give us \(n\) sample moment conditions that determine those parameters.