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Online Appendix to paper by Matthew Rognlie

Description of alternative procedure in section 3.2 for estimating path of r.

I sketch here the procedure in section 3.2 for estimating the effective required return $r(t)$ on capital for the US corporate sector. I specify the model in continuous time, and use superscripts to denote the time t for economy of notation. I am also more explicit here about how the underlying continuous time flows are aggregated into the measured flow for a given time period.

Assumed stochastic processes

I assume two stochastic processes beyond what is already visible in the data:

- π^t, which is a stationary, ergodic process for the share of gross output that goes to profits.
- ζ^t, which reflects stochastic pricing error for the total market value of corporations, with mean 1 (where 1 corresponds to no error).

Core relations

First relation: profit share of flows. We know that all non-profit income will be allocated between depreciation, labor, and the various types of capital. This is a flow relation

$$(1-\pi^t)Y^t = w^t L^t + \sum_i (\delta_i + r^t - g_{P_i}) P_i^t K_i^t$$

which can be rewritten as

$$\pi^t = 1 - \frac{w^t L^t}{Y^t} - \sum_i \frac{\delta_i P_i^t K_i^t}{Y^t} - \sum_i (r^t - g_{P_i}) \frac{P_i^t K_i^t}{Y^t} \tag{22}$$

or, if we don't want to divide by Y^t, as

$$\pi^t Y^t = Y^t - w^t L^t - \sum_i \delta_i P_i^t K_i^t - \sum_i (r^t - g_{P_i}) P_i^t K_i^t$$

Consolidating into an accumulated flow. Suppose that we write

$$\int_t^{t+\Delta t} \pi^s Y^s \, ds = \int_t^{t+\Delta t} (Y^s - w^s L^s - \sum_i \delta_i P_i^s K_i^s) \, ds - \sum_i \int_t^{t+\Delta t} (r^s - g_{P_i}) P_i^s K_i^s \, ds \tag{23}$$

We can identify the first part as simply real net capital income during the period, while for the second term we must write

$$\int_t^{t+\Delta t} (r^s - g_{P_i}) P_i^s K_i^s \, ds \approx \left((r^{t+\Delta t} - g_{P_i}) P_i^{t+\Delta t} K_i^{t+\Delta t} + (r^t - g_{P_i}) P_i^t K_i^t \right) \frac{\Delta t}{2} \tag{24}$$

Second relation: asset pricing. The expected discounted value of the profit stream from time t onward is (in real terms)

$$Y^t \cdot \int_0^\infty e^{(g_Y - \delta_\pi)s - \int_t^{t+s} r^u} \mathbb{E}_t[\pi^{t+s}]$$

where δ_π is the rate at which pure profits decay. (We can think of it as the rate at which a given company, for instance, on average loses the ability to make pure profits. There is no clear basis for picking δ_π, and I will choose $\delta_\pi = .015$, which implies a half-life of just below 50 years—within reason given the typical lifetimes of American corporations. Fortunately, the precise choice of δ_π does not matter much for the results.)

This expected discounted value plus the value of capital itself is (again in real terms)

$$Y^t \cdot \int_0^\infty e^{(g_Y - \delta_\pi)s - \int_t^{t+s} r^u du} E_t[\pi^{t+s}] \, ds + \sum_i P_i^t K_i^t$$

I assume that the market value of the corporate sector equals this overall value times ζ^t, the multiplicative stochastic pricing error that has mean 1, follows a stationary, ergodic process, and is drawn independently of π, $\{P_i^t\}$, $\{K_i^t\}$, and Y^t:

$$MV^t = \zeta^t \left(Y^t \cdot \int_0^\infty e^{(g_Y - \delta_\pi)s - \int_t^{t+s} r^u du} E_t[\pi^{t+s}] \, ds + \sum_i P_i^t K_i^t \right) \tag{25}$$

Define $OMV^t \equiv MV^t - \sum_i P_i^t K_i^t$, and rewrite (25) as

$$OMV^t = \zeta^t \left(Y^t \cdot \int_0^\infty e^{(g_Y - \delta_\pi)s - \int_t^{t+s} r^u du} E_t[\pi^{t+s}] \, ds \right) + (\zeta^t - 1) \sum_i P_i^t K_i^t \tag{26}$$

Second relation, part two: taking first differences. Now use (26) to compute

$$\frac{\phi(t)}{Y^{t-1,t}} \left(OMV^t - e^{-\delta_\pi \Delta t - \int_t^{t+\Delta t} r^u du} \cdot OMV^{t+\Delta t} \right) \tag{27}$$

for some small Δt, dividing by $Y^{t-1,t}$ (to be defined later, but known at time t) and multiplying by any deterministic function $\phi(t)$ of t. Expanding the term inside the parentheses in (27), we obtain

$$
\begin{aligned}
&OMV^t - e^{-\delta_\pi \Delta t - \int_t^{t+\Delta t} r^u du} OMV^{t+\Delta t} \\
&= \zeta^t \left(Y^t \int_0^\infty e^{(g_Y - \delta_\pi)s - \int_t^{t+s} r^u du} \mathbb{E}_t[\pi^{t+s}]\, ds \right) - \zeta^{t+\Delta t} \left(e^{-g_Y \Delta t} Y^{t+\Delta t} \int_{\Delta t}^\infty e^{(g_Y - \delta_\pi)s - \int_t^{t+s} r^u du} \mathbb{E}_{t+\Delta t}[\pi^{t+s}]\, ds \right) \\
&\qquad + (\zeta^t - 1) Y^t \sum_i P_i^t K_i^t + e^{-\delta_\pi \Delta t - \int_t^{t+\Delta t} r^u du} (\zeta^{t+\Delta t} - 1) Y^{t+\Delta t} \sum P_i^{t+\Delta t} K_i^{t+\Delta t} \qquad (28)
\end{aligned}
$$

Suppose now that we take the unconditional expectation of (27). Given the assumed independence of ζ^t, Y^t, and π^t, (28) simplifies dramatically and we are left with

$$
\mathbb{E}\left[\frac{\phi(t)}{Y^{t-1,t}} \left(OMV^t - e^{-\delta_\pi \Delta t - \int_t^{t+\Delta t} r^u du} \cdot OMV^{t+\Delta t} \right) \right] = \mathbb{E}\left[\frac{\phi(t)}{Y^{t-1,t}} Y^t \int_0^{\Delta t} e^{(g_Y - \delta_\pi)s - \int_t^{t+s} r^u du} \pi^{t+s}\, ds \right] \qquad (29)
$$

We can further manipulate (29), using the law of iterated expectations to obtain

$$
\begin{aligned}
\mathbb{E}\left[\frac{\phi(t)}{Y^{t-1,t}} Y^t \int_0^{\Delta t} e^{(g_Y - \delta_\pi)s - \int_t^{t+s} r^u du} \pi^{t+s}\, ds \right] &= \mathbb{E}\left[\frac{\phi(t)}{Y^{t-1,t}} \mathbb{E}_t \left[\int_0^{\Delta t} (e^{g_Y s} Y^t) e^{-\delta_\pi s - \int_t^{t+s} r^u du} \pi^{t+s}\, ds \right] \right] \\
&= \mathbb{E}\left[\frac{\phi(t)}{Y^{t-1,t}} \mathbb{E}_t \left[\int_0^{\Delta t} e^{-\delta_\pi s - \int_t^{t+s} r^u du} \pi^{t+s} Y^{t+s}\, ds \right] \right] \\
&= \mathbb{E}\left[\frac{\phi(t)}{Y^{t-1,t}} \int_0^{\Delta t} e^{-\delta_\pi s - \int_t^{t+s} r^u du} \pi^{t+s} Y^{t+s}\, ds \right] \qquad (30)
\end{aligned}
$$

Assuming that Δt is small enough and $\pi^{t+s} Y^{t+s}$ is sufficiently close to being continuous, we can approximate the integral inside (30) by

$$
\int_0^{\Delta t} e^{-\delta_\pi s - \int_t^{t+s} r^u du} \pi^{t+s} Y^{t+s} ds \approx \frac{1 + e^{-\delta_\pi \Delta t - (r^t + r^{t+\Delta t})/2}}{2} \int_0^{\Delta t} \pi^{t+s} Y^{t+s} ds \qquad (31)
$$

Full estimation strategy. We have shown that the unconditional expectation of (27), which we can approximate by

$$
\mathbb{E}\left[\frac{\phi(t)}{Y^{t-1,t}} \left(OMV^t - e^{-\delta_\pi \Delta t - (r^t + r^{t+\Delta t})/2} OMV^{t+\Delta t} \right) \right] \qquad (32)
$$

has unconditional expectation approximately equal to

$$
\mathbb{E}\left[\frac{\phi(t)}{Y^{t-1,t}} \cdot \frac{1 + e^{-\delta_\pi \Delta t - (r^t + r^{t+\Delta t})/2}}{2} \cdot \int_0^{\Delta t} \pi^{t+s} Y^{t+s} ds \right] \qquad (33)
$$

where according to (23) and (24), we can obtain the approximate flow of pure profits $\int_0^{\Delta t} \pi^{t+s} Y^{t+s} ds$ in (33) as

$$\begin{aligned} &\int_0^{\Delta t} \pi^{t+s} Y^{t+s}\, ds \\ &\approx \int_t^{t+\Delta t} (Y^s - w^s L^s - \sum_i \delta_i P_i^s K_i^s)\, ds - \sum_i \left((r^{t+\Delta t} - g_{P_i}) P_i^{t+\Delta t} K_i^{t+\Delta t} + (r^t - g_{P_i}) P_i^t K_i^t \right) \frac{\Delta t}{2} \end{aligned} \tag{34}$$

where the first term is just the recorded net return on capital in the period $[t, t + \Delta t]$ as measured in the national accounts, while the second term can be derived from the nominal quantities $P_i K_i$ of each type of capital.

The moments (32) and (33) are equal, and we can set the corresponding sample moments equal to each other. Generally I will look at an annual frequency, such that $\Delta t = 1$. Given a functional form for r^t with n free parameters to be pinned down, we can choose n functions for $\phi(t)$ to give us n sample moment conditions that determine those parameters.