Complementarity and Social Networks

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Abstract

I investigate complementarity games played on graphs, which model negative externalities embedded in structures of interaction. On the complete graph, the traditional economic analysis applies: the number of agents playing one strategy is proportional to its payoff. I show that, in general and contrary to coordination games, the structure crucially influences the equilibria. On an important class of graphs, called bipartite graphs, the equilibria do not depend on strategies' payoffs. On certain highly asymmetric graphs, an increase in the payoff of a strategy even decreases the number of agents playing this strategy. In most cases, equilibria do not maximize welfare.

Keywords: complementarity games, negative externalities, interaction structure, social networks.

JEL classification: C72, D62, Z13

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1 Introduction

In this paper, I analyze complementarity games played on graphs. Complementarity games model negative externalities, i.e., an increase in the number of agents playing one action decreases the relative payoff of this action. Games played on graphs model local and social interactions, i.e., each agent only interacts with a subset of the whole population. Hence, complementarity games played on graphs model situations where a negative externality is embedded in a structure of interaction. To illustrate such situations, let me present three stylized examples.

First, consider international trade and specialization. Assume that countries have to specialize in producing either primary or secondary commodities, and that trade mostly occurs between countries having different specializations. If costs associated with distance are high, two countries trade only when they are geographic neighbors. One activity, say producing secondary commodities, may be a priori more profitable than the other activity. Nonetheless, the more countries specialize in secondary commodities, the higher the relative payoff of specializing in primary commodities.

Second, consider localized pollution. Assume that cities use a clean or a dirty technology to produce a good, say transportation. The dirty technology generates pollution that partly diffuses to the neighboring environment. In isolation, a city prefers to use the dirty technology. However, the social cost associated with pollution is convex. Higher levels of ambient pollution increase the damages induced by an additional amount of pollution, thus increases the cost of using the dirty technology. The more the other cities use the dirty technology, the lower the incentive to use it.

Third, consider goods related to social distinctiveness. Assume that society is composed of nonconformists. Clothing may be formal or casual, and partly defines people's identity. Since individuals are non conformists, they want to differentiate themselves from their acquaintances. Everybody may have a different set of people to whom they compare. The more people dress formally, the higher the relative utility of dressing casually.[1]

[1] "Snob" goods are goods for which the demand decreases when others demand more of it, see Lebenstein

These three examples share two common aspects. The more the others choose an action, the lower the payoff of this action relative to the other action. Also agents do not interact uniformly with everybody else. The structure of interaction may be geographic, as in the case of trade and pollution, or social, as for clothing. The central question addressed by the paper is the following. When a negative externality is embedded in a structure of interaction, how do agents' choices depend on the payoffs and on the structure? Does the structure of interaction marginally change the traditional analysis of negative externalities or fundamentally modify it?

The central conclusion of the paper is that structure fundamentally modifies the economic analysis of negative externalities. As a first illustration of the results, let me contrast the outcomes obtained on two very different graphs: the complete graph and the star. The complete graph models traditional situations where everybody interacts with everybody else. On the complete graph in equilibrium, the number of people playing one strategy is simply proportional to the relative payoff of the strategy. If a strategy's payoff increases, the number of agents playing this strategy increases proportionally. On the contrary, the star is a centered structure: one agent, called the center, is linked with all the other agents, and all the other agents, called border agents, are only linked with the center. On the star in equilibrium, all the border agents have to play the strategy opposite from what the center plays. This is independent of the payoffs of the game.

Hence, on certain structures, choices of the agents do not depend on the actions' payoffs, but only on the structure itself. The star is not an isolated example, and I characterize the class of graphs on which this property holds. These graphs, called bipartite graphs, are the graphs on which everybody can be different from all their neighbors. On bipartite graphs, the effect of the structure is sufficiently strong to completely remove the influence of the payoffs. On certain other graphs, the structural effect may even be so strong that it reverses the payoffs' effect. I present a specific class of graphs on which an increase in the relative payoff of one strategy decreases the number of agents playing this strategy. These graphs

(1950). Desire for individual differentiation plays a critical role in fashion, e.g., Simmel (1904), Benvenuto (2000).

illustrate what may happen on highly asymmetric structures, in which few agents possess numerous connections while most agents have few connections.

This analysis relates to a growing literature concerned with the interplay between economic incentives and social structure. As done here, one strand of this literature relies on the assumption that interaction structure is fixed and frames the choices of the agents.[2] On the one hand, interaction structure may convey information between agents. Montgomery (1991) investigates incentives that firms have to hire through referral networks, in the presence of adverse selection on workers' ability. Bala and Goyal (1998) study bayesian learning occuring through graphs. Chwe (2000) analyzes how communication networks influence collective action.

On the other hand, and as done in this paper, the interaction structure may enter directly into agents' utilities. A lot of attention has been paid to coordination games played on graphs. Morris (2000) analyzes the prospect of contagion on graphs with infinite number of agents and bounded number of links per agent. Coordination games played on graphs have especially been studied within evolutionary game theory. In most settings, when everybody interacts with everybody, all agents eventually coordinate on the risk-dominant strategy, e.g., Young (1993), Kandori et al. (1993), Blume (1999). This result generalizes to regular local interaction structures, see Ellisson (1993), Blume (1993, 1995), and even to arbitrary symmetric graphs, see Young (1998). The structure of interaction affects the Nash equilibria of the game and the pace of convergence of the evolutionary process, but generally not the stochastically stable state. Even when interacting through a graph, everybody eventually coordinate on the risk-dominant strategy.[3]

[2] Another branch of this literature analyzes the incentives agents have to form and sever links with others, and the properties of emerging structures, e.g., Jackson and Watts (1997), Bala and Goyal (2000), Kranton and Minehart (1998), see also conclusion.

[3] Bergin and Lipman (1996) showed that in general, stochastic stability was highly dependent on the tremble process. Nonetheless, Blume (1999) showed that coordination on the risk-dominant strategy was the only stochastically stable state for a large class of trembles, when everybody interacts with everybody. On arbitrary graphs, this is true for log-linear trembles, see Young (1998), but not for uniform trembles, e.g., Jackson and Watts (1999).

I provide the counterpart of the analysis of coordination games played on graphs for complementarity games played on graphs. In my analysis, I use the framework developed by game theorists to study coordination games played on graphs, as presented in chapter 6 of Young (1998). The stage game is a 2 by 2 symmetric game. Complementarity games have two pure strategy Nash equilibria, in which one agent plays one strategy and the other agent plays the other strategy. Agents are linked through a symmetric unweighted graph, representing the structure of interaction. Agents play the 2 by 2 complementarity game with all the other agents with whom they are linked, and earn the sum of the payoffs of all these plays.[4] The resulting game is a n by n game depending on the 2 by 2 game and on the graph.

My central goal is to study how the Nash equilibria of this resulting game are determined by the payoffs of the complementarity game and by the graph. As for coordination games, complementarity games played on graphs usually have numerous Nash equilibria. This raises a serious problem of equilibrium selection. However, the game possesses an exact potential function in the sense of Monderer and Shapley (1996). Maxima of the potential are salient equilibria of the game, notably because they are the stochastically stable states for log-linear tremble, see Blume (1993), Young (1998). Therefore, I focus on potential maximization to study the equilibria of the game.

In the economic literature, little attention has been paid to complementarity games. Schelling (1978) early provided rigorous analysis of negative externalities. Complementarity games when everybody interacts with everybody have been analyzed by Kandori et al. (1993, section 6), and Canning (1995, section 5). Blume (1993, p.398), notices that on the infinite line, some Nash equilibria of a certain complementarity game are insensitive to variations in payoffs. All these results are encompassed and generalized in my analysis.[5]

[4] A key assumption is that agents play the same strategy with all their social neighbors. This assumption is standard and natural in the present context.

[5] Theoretically, complementarity games played on graphs are related to spin glasses systems from statistical physics with pure antiferromagnetic interaction. The central notion characterizing systems with antiferrromagnetic interaction is what physicists call 'frustration'; see Stein (1988) for an introduction to spin glasses systems.

In the preliminary section 2, I describe two by two complementarity games and the type of economic and social situations they allow one to model. In section 3, I pose the model for general symmetric two by two games played on symmetric graphs. I state the potential property and the results obtained for coordination games. In section 4, I analyze properties of the equilibria for complementarity games played on graphs. This section constitutes the core of the paper. I show that the equilibria crucially depend on the graph of interaction and investigate this dependence. I conclude in section 5.

2 Two by two complementarity games

In this preliminary section, I give stylized examples of complementarity games. Complementarity games are symmetric two by two games possessing two Nash equilibria in which agents play different strategies. Complementarity games are related to two broad types of economic and social situations: complementary production and exploitation.

First, complementarity games model situations where the joint production of a certain output requires that the agents adopt complementary roles or specializations. For example, countries might decide on how to specialize their economy in a way complementary to what their trading partners do[6]. Suppose that there are two countries, who can specialize in the production of raw materials or manufactured commodities. Opportunities for trade are greatest when they choose different specializations. The country that produces manufactured commodities uses raw materials imported from the other country as the fundamental inputs of its production. In turn, the country producing raw materials imports cheap manufactured goods from the other country. Even when countries have similar a priori characteristics, they might have an incentive to differentiate their specializations. An example of complementarity game that could be used to model this effect is as follows.

[6] In all these examples complementarity games only represent extremely stylized versions of the real processes involved. In some sense, complementarity games provide the simplest abstract models allowing one to represent complementary production and exploitation.

	Raw materials	Manufactured goods
Raw materials	1, 1	3, 5
Manufactured goods	5, 3	2, 2

In equilibrium both agents need not receive the same profit. Manufactured goods may be more profitable than raw materials and exchange is generally unequal. To some extent, the desire for social differentiation provides another example of complementary production. Consider the CEOs of two rival companies. They dislike each other and seek to define opposing images of their companies. Especially, they want to be dressed differently. If clothing can be casual or formal, this situation could be modelled with the following simple complementarity game

	Casual	Formal
Casual	0, 0	1, 1
Formal	1, 1	0, 0

In both examples, agents prefer to be in equilibrium. However, in other complementarity games one agent prefers to be off equilibrium. These games are related to situations of exploitation, congestion, and conflict. Consider for example two neighboring cities that have to decide how to produce electricity. They can use a cheap but polluting technology (e.g. burning coal) or an expensive and clean one (e.g. solar energy). Pollution is global and pollution damages are convex. Suppose that the polluting technology is costless, while the clean technology costs 3. Suppose that pollution damages to each city are 2 when only one technology uses the polluting technology, but 6 when both cities use the polluting technology. Finally, the profit of electricity production is equal to 6. The resulting payoffs are

	Polluting	Clean
Polluting	0, 0	4, 1
Clean	1, 4	3, 3

This is a complementarity game, in which the equilibria do not Pareto dominate coordination on Clean.[7] The global welfare, defined as the sum of the payoffs, is even highest when both cities choose the Clean technology. This game is akin to the Hawk-Dove game studied by biologists, see Hofbauer and Sigmund (1998, ch. 6.1). In some sense, the Polluting city exploits the Clean city. In summary, complementarity games represent situations of complementary production and exploitation.

3 Symmetric 2 by 2 games played on symmetric graphs

3.1 The model

Consider a symmetric 2 by 2 game Γ defined by strategies $\{A, B\}$ and payoff u. Such a game has a general payoff structure

$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \left[\begin{array}{cc} a,a & c,d \\ d,c & b,b \end{array}\right] \end{array}$$ or equivalently, $$\begin{array}{ccc} u & A & B \\ A & a & c \\ B & d & b \end{array}$$ where u denotes the payoff of Column.

Consider a society of n agents, denoted by i. Agents are embedded in a social network, modelled as a symmetric unweighted graph g. The link between agent i and agent j is denoted by g_{ij}. The symmetry of g means that $\forall i, j, g_{ij} = g_{ji}$. The assumption that g is unweighted means that $\forall i, j, g_{ij} \in \{0, 1\}$. Agent i and agent j are **social neighbors** when $g_{ij} = 1$. Moreover, I assume that $\forall i, g_{ii} = 0$. There is no private utility, agents only get utility from interacting with others.[8] Finally, I assume that there is no isolated individual,

[7] In general, if c denotes the cost of the Clean technology, C_1 the pollution cost when only one city uses the Polluting technology and C_2 the pollution cost when both cities use the Polluting technology, the resulting game is one of complementarity iff

$$C_1 < c < C_2 - C_1$$

Welfare is highest when both cities choose the Clean technology iff

$$c < 2C_1$$

[8] All these assumptions are discussed in conclusion.

everyone is connected with someone else.

Each agent i chooses a strategy $x_i \in \{A, B\}$. Let $x = (x_1, ..., x_n) \in \{A, B\}^n$ be the vector of strategies, called the state of the system. Each agent plays the game Γ with all his social neighbors, and earns the sum of the utilities of all these plays. I denote by $u_i(x)$ the utility of agent i. By definition,

$$u_i(x) = \sum_{j=1}^{n} g_{ij} u(x_i, x_j) = \sum_{j: g_{ij}=1} u(x_i, x_j)$$

Payoffs u_i define a n player game, denoted by Γ_g, depending on the 2 player game Γ and on the graph of interaction g. In short, Γ_g is the game Γ played on the graph g.

Given a strategy vector x and a graph g, I denote by n_A the number of agents playing A, by n_i the number of social neighbors of i, by $n_i(A)$ the number of neighbors of i playing A, by $n(AA)$ the number of links between agents playing A, and by $n(AB)$ the number of links connecting one agent playing A and one agent playing B. Similar definitions apply to n_B, $n_i(B)$, and $n(BB)$. Abusing notation, define $A = \{i \in [1, n] : x_i = A\}$ and $B = \{i \in [1, n] : x_i = B\} = [1, n] \backslash A$. Equivalently,
$n_i = \sum_j g_{ij}$ and $n_i(A) = \sum_{j \in A} g_{ij}$
$n(AA) = 1/2 \sum_{i,j \in A} g_{ij}$ and $n(AB) = \sum_{i \in A, j \in B} g_{ij}$
Note the straightforward following relations:
$n_A + n_B = n$ and $n_i(A) + n_i(B) = n_i$
$n(AA) = 1/2 \sum_{i \in A} n_i(A)$ and $n(AB) = \sum_{i \in A} n_i(B) = \sum_{i \in B} n_i(A)$
$n(AA) + n(BB) + n(AB) = |g|$ the total number of links of the graph g.

3.2 The potential function

Because of the symmetric form of the game and the graph, Γ_g is an exact potential game in the sense of Monderer and Shapley (1996). This means that all the individual utilities are aligned with the same 'collective' function, called potential, which provides a powerful device to study the equilibria of the game.

Proposition 3.1 (Young, 1998) *For any symmetric two by two game* Γ *and symmetric graph* g*, the* n *player game* Γ_g *is an exact potential game with potential function*

$$\varphi(x) = (a-d)n(AA) + (b-c)n(BB)$$

Proof: $u_i(A, x_{-i}) = an_i(A) + cn_i(B)$ and $u_i(B, x_{-i}) = dn_i(A) + bn_i(B)$
$u_i(A, x_{-i}) - u_i(B, x_{-i}) = (a-d)n_i(A) - (b-c)n_i(B)$
Compare this difference in u_i with the difference in the potential:
$\varphi(A, x_{-i}) - \varphi(B, x_{-i}) = (a-d)[n(AA)(A, x_{-i}) - n(AA)(B, x_{-i})] - (b-c)[n(BB)(B, x_{-i}) - n(BB)(A, x_{-i})]$
Since the graph is symmetric
$n(AA)(A, x_{-i}) - n(AA)(B, x_{-i}) = n_i(A)$ and similarly,
$n(BB)(B, x_{-i}) - n(BB)(A, x_{-i}) = n_i(B)$.
Therefore $\forall i, x_{-i}, u_i(A, x_{-i}) - u_i(B, x_{-i}) = \varphi(A, x_{-i}) - \varphi(B, x_{-i})$. QED.

As for any potential game, Nash equilibria of Γ_g are local maxima of the potential and vice versa. The potential function contains all the informations needed to describe the equilibria of the game. Note that the three games

$$\begin{bmatrix} a,a & c,d \\ d,c & b,b \end{bmatrix}, \begin{bmatrix} a-d, a-d & 0,0 \\ 0,0 & b-c, b-c \end{bmatrix} \text{ and } \begin{bmatrix} 0,0 & c-b, d-a \\ d-a, c-b & 0,0 \end{bmatrix}$$

have the same potential function. Nash equilibria of Γ_g only depend on the relative payoffs of Γ, even if welfare properties of these three games may differ.

The potential gives a natural way to reduce the multiplicity of Nash equilibria. Maxima of the potential are stochastically stable states of the myopic best-reply process of Γ_g under the log linear perturbation, see Appendix 1. Therefore, maxima of the potential are salient equilibria of the game and I will focus my analysis on them.

Finally, note that an agent with more connections (high n_i) has more influence on the potential, hence on the equilibria. This happens because agents' choices enter the potential function through the numbers of links $n(AA)$ and $n(BB)$. This property is important, and may be crucial to understand certain counterintuitive results, see Example 4.1 below.

3.3 Coordination games

I now briefly restate the results obtained for coordination games. A symmetric two by two game is a coordination game if $a > d$ and $b > c$. Or equivalently if pure strategy strong Nash equilibria of Γ are (A, A) and (B, B). Notice that $u_i(A, x_{-i}) > u_i(B, x_{-i}) \Leftrightarrow n_i(A) > pn$ where $p = (b - c)/(a - d + b - c)$. Agents want to play A when a sufficient proportion of their neighbors play A.

In general, Γ_g may have numerous Nash equilibria, but has only one highest potential equilibrium. If two subgroups of agents have high intra group connections and comparatively low inter group connections, everybody playing A in the first subgroup and everybody playing B in the second subgroup can be sustained as an equilibrium. Both strategies coexist when the graph naturally splits in two groups. However, there is only one equilibrium maximizing the potential: everybody plays the risk dominant strategy, see Young (1998). This result holds for any graph of interaction g. To see this, note that for coordination games $a - d$ and $b - c$ are positive. Suppose that A is risk dominant, i.e., $a - d > b - c$.

$$\varphi(x) = (a - d)n(AA) + (b - c)n(BB) \leq (a - d)(n(AA) + n(BB)) \leq (a - d)|g|$$

This expression is an equality if and only if $n(BB) = n(AB) = 0$, which means that everybody plays A. Therefore, everybody playing the risk-dominant strategy is the unique global maximum of the potential for coordination games.

4 Complementarity games

Suppose that Γ is a complementarity game, i.e., $a < d$ and $b < c$. Equivalently, pure strategy strong Nash equilibria of Γ are (A, B) and (B, A). Notice that $u_i(A, x_{-i}) > u_i(B, x_{-i}) \Leftrightarrow n_i(B) > (1 - p)n_i$. Agents want to play A when a sufficient proportion of their neighbors play B.[9] In this section, I analyze how the equilibria of Γ_g depend on the game Γ and the

[9] There are natural symmetries between complementarity games played on graphs and coordination games played on graphs, that I explore in Appendix 2.

graph g. A central result is that, contrary to coordination games, the graph g crucially determines the highest potential equilibria.

Let me introduce an additional notation. As defined above, $p = (c-b)/(c-b+d-a)$ is the probability of playing A in the mixed Nash equilibrium of Γ, i.e., p is the probabiltiy of playing A that makes the other player indifferent between the two strategies. I define $M = (c-b)/(d-a) = p/(1-p)$ as the ratio of the relative payoff of playing A against B over the relative payoff of playing B against A. $M > 1$ is equivalent to $p > 1/2$ and means that playing A against B is preferred to playing B against A. In short, A is preferred to B.

I develop my analysis as follows. First, I describe the equilibria for the complete graph. This case corresponds to the traditional economic situation with decreasing returns when everybody interacts with everybody. Second, I introduce bipartite graphs and show that the equilibria on bipartite graphs do not depend on the payoffs of the game. Third, I investigate the situation for general graphs. Maximization of the potential is related to a well-known hard combinatorial optimization problem called MAX CUT. I characterize the set of Nash equilibria when the relative payoff of one strategy is much higher than the payoff of the other strategy. I provide comparative statics analysis for fixed graphs when payoffs evolve and give examples of graphs on which an increase in the benefit of playing A actually **decreases** the number of people playing A. Fourth, I provide a welfare analysis of the game. In most cases, highest potential equilibria do not maximize welfare. Fifth and finally, I study the mixed equilibria of the game. I characterize them and show that they do not maximize the potential.

4.1 Complete graph

Consider the complete graph, i.e., $\forall i \neq j, g_{ij} = 1$. The complete graph models situations where everybody interacts uniformly with everybody else.

PROPOSITION 4.1 *On the complete graph, Nash equilibria of Γ_g are states where the proportion of agents playing A is 'almost' equal to the mixed equilibrium probability p. All these equilibria are global maxima of the potential.*

PROOF: Suppose that k players play A and $n-k$ players play B. An agent playing A earns an utility of $(k-1)a+(n-k)c$. If he changed to B, he would earn $(k-1)d+(n-k)b$. Agents playing A play a best response if and only if $(k-1)a+(n-k)c \geq (k-1)d+(n-k)b$. Similarly, agents playing B play a best response if and only if $kd+(n-k-1)b \geq ka+(n-k-1)c$. Therefore, a state with k players playing A is an equilibrium if and only if

$(n-k)(c-b) \geq (k-1)(d-a)$ and $(n-k-1)(c-b) \leq k(d-a)$

This is equivalent to $(n-1)p \leq k \leq (n-1)p+1$ and to

$$-p/n \leq k/n - p \leq (1-p)/n$$

There is at least one solution and at most two. There are two solutions if and only if $(n-1)p$ is an integer. When there is only one solution, nobody is indifferent and equilibria are strong. When there are two solutions, equilibria are weak: A players are indifferent in one solution, and B players are indifferent in the other.

Alternatively, we can use the potential:

$2\varphi(x) = (a-d)k(k-1) + (b-c)(n-k)(n-k-1)$

$2\varphi(x) = -k^2(d-a+c-b) + k[2n(c-b)+d-a-(c-b)] - n^2(c-b)$

As a function on the whole interval $[0,n], \varphi$ is concave and takes its unique maximum value at

$k^* = n(c-b)/(d-a+c-b) + 1/2[d-a-(c-b)]/[d-a+c-b] = (n-1)p+1/2$

Maximum values of φ for integers are attained for the integer(s) closest to k^*. This yields the same solution as above. Moreover, if there are two closest integers, they have the same potential. Thus, all the equilibria have the same potential. QED.

For instance, when p is close to 1 (specifically, $p > 1 - 1/n$), the relative payoff of A is much higher than the relative payoff of B. In this case, Nash equilibria are states with one agent playing B and all the other agents playing A (see also Proposition 4.3). Note that the state where all the agents play one strategy is never a Nash equilibrium of a complementarity game played on a graph. If all the neighbors of an agent play A, this agent has an incentive to play B, independently of the payoffs.

When $p = 1/2$, A and B have the same relative payoff. If n is even, equilibria are states with $n/2$ players playing A and $n/2$ players playing B. These equilibria are strong. If n is odd, equilibria are states with $(n-1)/2$ players playing one strategy and $(n+1)/2$ players playing the other strategy. These equilibria are weak: players playing the majority strategy are indifferent.[10]

Proposition 4.1 is similar to theorem 5 in Kandori et al. (1993), and to the classical result in biology for a population playing the Hawk-Dove game, e.g. Hofbauer and Sigmund (1998, ch.6.1). This result corresponds to the usual economic situation of global decreasing returns. Agents reach their indifference point between the two strategies. An increase of the relative payoff of A with respect to the relative payoff of B translates into a proportional increase of the number of people playing A. How does this traditional economic intuition apply when agents are embedded in a social network?

4.2 Bipartite graphs

A first element of the answer is that there is a general class of graphs, called bipartite graphs, on which highest potential equilibria do not depend on the payoffs of the complementarity game. First, let us define bipartite graphs, e.g., Bondy and Murty (1976), ch.8.

DEFINITION 4.1 *A graph is bipartite if there is a partition (A, B) of the set of agents such that all the links occur between A agents and B agents. Such a partition is then called a bipartition of the graph.*

In other words, a graph is bipartite if there is a vector of strategy x such that $n(AA) = n(BB) = 0$. This implies that for every complementarity game $\Gamma, \varphi(\Gamma, g, x) = 0$. Reciprocally, if there is a complementarity game $\hat{\Gamma}$ and a vector of strategies x such that $\varphi(\hat{\Gamma}, g, x) = 0$, it implies that $n(AA) = n(BB) = 0$, and the graph is bipartite. I now state the property of bipartite graphs.

[10] A noteworthy consequence is that if n is odd, the myopic best-reply process P under the randomized tie-breaking rule is ergodic, see Appendix 1.

Proposition 4.2 *Bipartite graphs are the graphs for which highest potential equilibria do not depend on the payoffs of* Γ. *When a bipartite graph is connected, it has two highest potential equilibria symmetric to one another.*

Proof: If the graph is bipartite, the potential is maximized for $n(AA) = n(BB) = 0$. This occurs for any bipartition of the graph, which do not depend on the payoff of the complementarity game. Reciprocally, suppose that highest potential equilibria do not depend on the payoffs of Γ. Consider a game for which $M > n - 1$. Proposition 4.3 (see below) implies that there cannot be a BB link in equilibrium, hence in highest potential equilibria. Now consider a game for which $1/M > n - 1$. This time, there cannot be a AA link in equilibrium. Therefore $n(AA) = n(BB) = 0$ and the graph is bipartite. Highest potential equilibria are the bipartitions of bipartite graphs. When a bipartite graph is connected, it has only two bipartitions: (A, B) and (B, A). QED.

On bipartite graphs, the number of people playing A does not depend on the payoffs of Γ. This number is a structural parameter that only depends on the graph. Some bipartite graphs are symmetric ($n_A/n \approx 1/2$), and some are highly asymmetric ($n_A/n \approx 0$ and 1). Numerous important graphs are bipartite, including stars, trees and lattices with nearest neighbor interaction. Bipartite graphs are the graphs on which everybody can be different from all their neighbors. When a graph is bipartite, two agents linked together cannot be both linked to a common third agent. (In other words, bipartite graphs do not have triangles). Bipartite graphs appear naturally in situations where agents are divided in two groups, for example buyers and sellers, or men and women.

4.3 Some insights on the general situation

Proposition 4.2 shows that the graph of interaction plays a crucial role for complementarity games. On bipartite graphs, equilibria do not depend on payoffs of Γ. On the complete graph, equilibria are proportionally related to payoffs of Γ. Most graphs are neither bipartite, nor complete. What can be said for the general case?

In this section, I show that finding the maximum of the potential for an arbitrary graph is a hard problem unlikely to lead to closed form solutions. To obtain insights, I specify the question in three directions. How to maximize the potential when relative payoffs are equal? How to maximize the potential when one strategy is highly preferred to the other strategy? For a given graph, how do maxima of the potential change when relative payoffs change?

First, when relative payoffs are equal, maximizing the potential is equivalent to finding the minimum number of links that have to be severed from g to obtain a bipartite graph. Equivalently, highest potential equilibria are bipartitions of maximal bipartite subgraphs of g. To see this, notice that $\max_{x\in\{A,B\}^n} \varphi(\Gamma, g, x) = -(d-a) \min_{x\in\{A,B\}^n}[n(AA)+Mn(BB)]$. When $M = 1$ maximizing the potential is equivalent to minimizing $n(AA) + n(BB)$, which is the number of links between similar agents. The graph obtained from g by severing these links is bipartite. Therefore, maximizing the potential is equivalent to finding the minimum number of links that have to be severed from g to obtain a bipartite graph. This number $\min_{x\in\{A,B\}^n}[n(AA) + n(BB)]$ is a measure of how far a graph is from being bipartite, that I call the **frustration** of the graph.[11] Moreover, remember that $n(AA) + n(BB) = |g| - n(AB)$. Thus, maximizing the potential is equivalent to maximizing $n(AB)$, which is the number of links of the bipartite subgraph of g induced by the partition (A, B). In other words, highest potential equilibria correspond to bipartitions of maximal bipartite subgraphs of g.

Reformulating maximization of the potential in graph theoretic terms should allow us to rely on results from graph theory to characterize highest potential equilibria. In fact, finding the maximal bipartite subgraphs of a graph is a well known problem of combinatorial optimization called MAX CUT (with my notation, the 'cut' of the graph g by the partition A, B is $n(AB)$). A fundamental finding of computer scientists is that MAX CUT is a NP-complete problem, see Garey and Johnson (1979).[12] For our purposes, the fact that

[11]Note that frustration and more generally $\min_{x\in\{A,B\}^n}[n(AA) + Mn(BB)]$ with $M > 0$ are increasing with respect to graph inclusion, hence are maximal for the complete graph.

[12]In a few words, a problem is NP-complete when there is no algorithm that solves the problem in polynomial time (unless P=NP). MAX CUT is solvable in polynomial time for planar graphs, see Hadlock

MAX CUT is NP-complete indicates that we are not likely to be able to obtain closed form descriptions of the highest potential equilibria in general.[13]

Second, when the relative payoff of one strategy is much higher than the relative payoff of the other strategy, I obtain a simple characterization of the Nash equilibria of Γ_g.

PROPOSITION 4.3 *Suppose that $M > n-1$, so that strategy A is highly preferred to strategy B. Nash equilibria of Γ_g are the states in which every agent is connected to an agent playing B and no two agents playing B are connected.*[14]

PROOF: Notice that $u_i(A, x_{-i}) > u_i(B, x_{-i}) \Leftrightarrow n_i(A) < Mn_i(B)$. Suppose agent i has at least one neighbor playing B: $n_i(B) \geq 1$. This implies that $n_i(A) \leq (n-1) < M \leq Mn_i(B)$, and $u_i(A, x_{-i}) > u_i(B, x_{-i})$. An agent connected with another agent playing B always plays A. Suppose now that $n_i(B) = 0$. Then, $u_i(B, x_{-i}) - u_i(A, x_{-i}) = (d-a)n_i > 0$. When surrrounded only by agents playing A, one has an incentive to play B. Reciprocally, consider a state without BB link and such that every agent playing A is connected to an agent playing B. Agents playing B are only connected with agents playing A, hence do not have an incentive to switch. Agents playing A are connected to at least one agent playing B, hence do not have an incentive to switch. QED.

When $M > n-1$ in equilibrium, $n(BB) = 0$. Highest potential equilibria are the Nash equilibria with the minimum number of AA links. Proposition 4.3 provides a simple way to list all the Nash equilibria, when the relative payoff of one strategy is much higher than the relative payoff of the other strategy.

Third, I derive comparative statics result. For a given graph g, how do highest potential equilibria vary when the benefit of playing A increase?

(1975). For general graphs, relatively good 'approximation' algorithms have been designed, see Goemans and Williamson (1995).

[13]When $M \neq 1$, finding the minimum of $n(AA) + Mn(BB)$ is a problem that has not been studied by computer scientists, although is very probably NP-complete as well, Samir Khuller (2000, personal communication).

[14]In graph theoretic terms, the set of agents playing B is an 'independent' set of the graph g, maximal with respect to inclusion, see Bondy and Murty (1976, ch. 7).

A first result is that there are graphs on which an increase in the benefit of playing A actually decreases the number of agents playing A. To see this, define the graph $g_{n,k}$ as follows. In the center, n agents are connected to each other through a complete graph. In addition, each of these central agents is connected with k other agents through one by one links (see Appendix 3 for a picture of $g_{4,2}$). The total number of agents is $n + nk$ and the total number of links is $n(n-1)/2+nk$. In equilibrium, every 'border' agent has to play the opposite strategy from the central agent with whom he is linked. Hence the contribution of border agents to the potential is 0, and maximizing the potential on $g_{n,k}$ is equivalent to maximizing the potential for the complete graph with n agents. From Proposition 4.1, we know that the number of agents playing A in the center is almost pn. Each of these agents induce k border agents to play B. Therefore $n_A \approx pn + (1-p)nk = nk + p(n-nk)$. If $k = 1$, n_A is independent of the payoffs of Γ (even if the equilibria are not) and if $k > 1$, n_A decreases when p increases.

Example 4.1 *On $g_{n,k}$ when $k > 1$ the number of agents playing A in highest potential equilibria decreases when the benefit of playing A increases.*

How to understand this paradoxical result? $g_{n,k}$ is a particular graph in which most agents have few links and few agents have many links. Highly connected agents have more influence on the shape of the equilibria. On the subset of these highly connected agents, an increase in the payoff of A increases the number of agents playing A (center effect). However, every highly connected agent who plays A induces several poorly connected agents to play B (border effect). On the whole population, the border effect may overcome the center effect. Note that $g_{n,k}$ has a low frustration compared to the total number of agents. A low number of links determines the strategies played by the whole population.

Despite the previous example, one can still obtain regular monotonicity conditions. However, these conditions concern the links between agents instead of the agents themselves.

Proposition 4.4 *When M increases, $n(AA)$ increases and $n(BB)$ decreases. If $M' > M > 1$, $n(AB)$ increases and if $M < M' < 1$, $n(AB)$ decreases.*

PROOF: Suppose that $M < M'$ and denote by x_M a solution of $\min_x[n(AA) + Mn(BB)]$. We have:

$n(AA)(x_M) + Mn(BB)(x_M) \leq n(AA)(x_{M'}) + Mn(BB)(x_{M'})$ and

$n(AA)(x_{M'}) + M'n(BB)(x_{M'}) \leq n(AA)(x_M) + M'n(BB)(x_M)$

This is equivalent to

$M'[n(BB)(x_{M'}) - n(BB)(x_M)] \leq n(AA)(x_M) - n(AA)(x_{M'}) \leq M[n(BB)(x_{M'}) - n(BB)(x_M)]$

Since $M < M'$, $n(BB)(x_{M'}) \leq n(BB)(x_M)$ and $n(AA)(x_{M'}) \geq n(AA)(x_M)$

Moreover, $n(AB) = \#g - n(AA) - n(BB)$, thus

$(M'-1)[n(BB)(x_{M'}) - n(BB)(x_M)] \leq n(AB)(x_{M'}) - n(AB)(x_M) \leq (M-1)[n(BB)(x_{M'}) - n(BB)(x_M)]$

If $M' < 1$, $n(AB)(x_{M'}) \geq n(AB)(x_M)$, and if $M > 1$, $n(AB)(x_{M'}) \leq n(AB)(x_M)$. QED.

Thus, when the relative payoff of a strategy increases, the number of links between agents playing this strategy increases and the number of links between agents playing the other strategy decreases, whereas the effect on the number of agents playing the strategy is ambiguous.

4.4 Welfare analysis

I now provide a welfare analysis of Γ_g. For coordination games, tension between maximization of the potential and Pareto optimality is clear. If the risk dominant equilibrium of Γ Pareto dominates the other equilibrium, highest potential equilibria of Γ_g are Pareto optimal, otherwise they are not. What is the counterpart of this tension for complementarity games? There are several ways to define the social welfare associated with a vector of strategies x in the game Γ_g. In this paper, I focus on a simple way. I define the welfare of a vector of strategies x, denoted $W(x)$, as the sum of the utilities

$$W(x) = \sum_{i=1}^{n} u_i(x)$$

States maximizing W are Pareto optimal, but the converse need not be true. Welfare is directly related to links of g. A link between two agents playing A generates for each of them a payoff of a. Therefore contribution of a AA link to welfare is $2a$. Similarly, a AB link generates a payoff of c for the agent playing A and d for the agent playing B. Contribution of a AB link to welfare is $c+d$.

LEMMA 4.1 $W(x) = 2an(AA) + (c+d)n(AB) + 2bn(BB)$

PROOF: $u_i(A, x_{-i}) = an_i(A) + cn_i(B)$, and $u_i(B, x_{-i}) = dn_i(A) + bn_i(B)$
Therefore, $W(x) = \sum_{i \in A}[an_i(A) + cn_i(B)] + \sum_{i \in B}[dn_i(A) + bn_i(B)]$
Moreover, $\sum_{i \in A} n_i(A) = 2n(AA)$ and
$\sum_{i \in A} n_i(B) = \sum_{i \in B} n_i(A) = n(AB)$. QED.

Using Lemma 4.1, one can see that welfare is associated with a certain potential function. Effectively, since $n(AA) + n(AB) + n(BB) = |g|$,

$$W(x) = (c+d)|g| + (2a - c - d)n(AA) + (2b - c - d)n(BB)$$

Therefore, $W(x) = (c+d)|g| + \varphi(\tilde{\Gamma}, g, x)$, where

$$\tilde{\Gamma} = \begin{bmatrix} 2a, 2a & c+d, c+d \\ c+d, c+d & 2b, 2b \end{bmatrix}$$

The game $\tilde{\Gamma}$ is derived from the game Γ when both agents act as a single unit whose payoff is the sum of individuals' payoffs. When playing $\tilde{\Gamma}$, agents fully internalize the effect of their decision on the global welfare. Welfare maxima of Γ_g are highest potential equilibria of $\tilde{\Gamma}_g$.[15] This property allows one to derive several results.

First, if Γ is a coordination game, maximizing welfare implies that when $2a - c - d > 2b - c - d \Leftrightarrow a > b$, everybody plays A, which is the standard result.

Second, note that $\tilde{\Gamma}$ is not always a complementarity game. When $(c+d)/2 < a$, $\tilde{\Gamma}$ admits (A, A) for unique Nash equilibrium, and the potential is maximized for everybody playing A (which is never a Nash equilibrium of a complementarity game played on a graph). This situation corresponds to the games of exploitation and conflict presented in section 2.

[15] This provides an additional justification to the study of the potential function and its maxima.

Third, on bipartite graphs highest potential equilibria do not depend on the payoffs. Thus, as soon as $\tilde{\Gamma}$ is a complementarity game, i.e., $\max(a,b) < (c+d)/2$, highest potential equilibria maximize welfare.

Fourth, the welfare function and the potential function are a priori aligned if and only if $c = d$. In this case, agents receive the same payoffs in every configuration and there is no externality.

Fifth, all the games with given a, b, and $c + d$ have the same welfare function. In this case, the greater $|c - d|$ the lower the welfare of the highest potential equilibria vis a vis maximal welfare. Proposition 4.4 shows that highest potential equilibria have too much and/or too few AA, BB, and AB links, depending on the values of a, b, c, and d.

In summary:

Proposition 4.5 *If $c = d$ or if the graph is bipartite and $\max(a,b) < (c+d)/2$, highest potential equilibria are the welfare maxima. If $a > (c+d)/2$, welfare is maximized for global coordination on A. In general, highest potential equilibria do not maximize welfare and the greater $|c-d|$, the lower the welfare of highest potential equilibria vis a vis maximal welfare.*

4.5 Mixed equilibria

In this section, I examine the mixed equilibria of Γ_g. So far, I have focused on pure strategy equilibria: agents played A or B, but did not play a random mix of both strategies. Perhaps surprisingly, mixed equilibria of Γ_g are much simpler to characterize than pure strategy equilibria. However, I show that no mixed equilibrium is a global maximum of the potential. This justifies the focus on pure strategy equilibria, at least from the point of view of potential maximization.

Suppose that agents play mixed strategies, and denote by $x_i \in [0,1]$ the probability of agent i to play A. An equilibrium of Γ_g is mixed if $\forall i, x_i \notin \{0,1\}$. The expected utility of agent i is

$$u_i(x) = \sum_j g_{ij}[ax_i x_j + cx_i(1-x_j) + d(1-x_i)x_j + b(1-x_i)(1-x_j)]$$

As usual, agent i has a possible interest to randomize if and only if he is indifferent between the two strategies, i.e., $\partial u_i / \partial x_i = 0$. This is equivalent to

$$(a-d)\sum_j g_{ij}x_j = (b-c)\sum_j g_{ij}(1-x_j)$$

The potential function is usually defined for pure strategies. For mixed strategies, I consider the expected value of the potential:

$$\varphi(x) = (a-d)\sum_{i,j} g_{ij}x_i x_j + (b-c)\sum_{i,j} g_{ij}(1-x_i)(1-x_j)$$

φ is simply a polynomial of second degree in x. The first order derivative of φ with respect to x_i is

$$\partial\varphi/\partial x_i = (a-d)\sum_j g_{ij}x_j - (b-c)\sum_j g_{ij}(1-x_j)$$

Therefore, x is a mixed equilibrium of Γ_g if and only if the first order derivatives of φ at x are equal to 0. Using matrix notations, this can be rewritten as $(a-d)gx = (b-c)g(1-x)$, where 1 denotes the vector of ones. This is equivalent to $(a+b-c-d)gx = (b-c)g1$ and to $g(x-p.1) = 0$. In other words, $x - p.1$ is an element of $Ker(g) = \{u \in R^n : gu = 0\}$.

Could these mixed equilibria be global maxima of the potential? One has to check the second order conditions

$$\partial^2\varphi/\partial x_i \partial x_j = (a+b-c-d)g_{ij}$$

Therefore, the Hessian of φ is proportional to g. The proportionality constant is positive for coordination games and negative for complementarity games. Remember that diagonal terms of g are equal to 0. This implies that $Tr(g) = 0$, hence g is neither a positive semi-definite matrix, nor a negative semi-definite matrix (except for $g = 0$, the empty graph). This means that maxima of φ cannot be interior solutions. To summarize these findings:

PROPOSITION 4.6 *x is a mixed equilibrium of Γ_g if and only if $\forall i, x_i \notin \{0,1\}$ and $x = p.1 + u, u \in Ker(g)$. No mixed equilibrium is a global maximum of the potential.*

Note that Proposition 4.6 is valid for complementarity games as well as coordination games.

5 Conclusion

As advocated by sociologists, e.g., Granovetter (1985), Burt (1995), and increasingly recognized by economists, e.g., Manski (2000), social structure frames economic outcomes. Increasing returns embedded in social networks underlie many economic and social situations. Game theorists have analyzed such positive externalities with coordination games played on graphs. The graph has a strong effect on transitory situations, i.e., Nash equilibria, but not on long run outcomes, i.e., equilibria of highest potential. In this paper, I have argued that decreasing returns embedded in social networks were equally important and much less studied. Such negative externalities model situations of complementarity, congestion, conflict, and nonconformism embedded in geographic or social structures. I have investigated complementarity games played on graphs and shown that the graph had a strong effect on long run outcomes.

I first give a summary of the results. On the complete graph, the number of agents playing one strategy is proportional to the relative payoff of this strategy. On bipartite graphs, highest potential equilibria do not depend on the payoffs of the complementarity game. On arbitrary graphs, if strategies have the same relative payoffs, highest potential equilibria correspond to bipartitions of maximal bipartite subgraphs. This is known to be a hard combinatorial optimization problem. When the relative payoff of one strategy is much higher than the relative payoff of the other strategy, Nash equilibria are the independent sets of the graph maximal with respect to inclusion. When the payoff of a strategy increases, the number of links between agents playing this strategy increases and the number of links between agents playing the other strategy decreases. This is not true for the agents themselves. On certain highly asymmetric graphs, an increase in the benefit of playing a strategy decreases the number of agents playing this strategy. In most cases, highest potential equilibria do not maximize welfare.

I now discuss some possible extensions of the model. In the model, only two strategies are available. How to generalize to $k \geq 3$ strategies? The first difficulty of this exercise would be to define what complementarity games are when the number of strategies is greater than 2. For k strategies, graphs on which everybody can be different from their neighbors are graphs of 'chromatic number' less than or equal to k, see Bondy and Murty (1976, ch.8). Therefore, graphs of chromatic number less than or equal to k are certainly the graphs on which (certain) equilibria are insensitive to variations in payoffs, hence would constitute the appropriate generalization of bipartite graphs.

My model is a model of pure negative externalities embedded in unweighted symmetric graphs and without private utilities. Agents could have private utilities $v_i(x_i)$ independent on choices of the others. Many models on social interactions assume that an individual's utility is the sum of a private and a social component, e.g., Durlauf (1997). Multiplicity of equilibria usually appears when the magnitude of the social component is large enough. Private utilities should decrease the influence of the structure.

Links could be weighted and/or negative. Negative links and coordination are equivalent to positive links and complementarity. Similarly, negative links and complementarity are equivalent to positive links and coordination. Note that existence of a potential is still guaranteed for weighted links and private utilities, as soon as links are symmetric, see Young (1998). Axelrod (1997) applies such a framework to two instances of coalition formation: the alignment of European nations in the Second World War (ch. 4), and competing UNIX operating system standards (ch. 5).

Another promising way to explore is the endogenization of the network. What happens when agents choose with whom they interact, rationally form and sever social links? Jackson and Watts (1999) and Goyal and Vega-Redondo (1999) study coordination games played on endogenous graphs. Two common conclusions of these two studies are that stochastically stable networks are (mostly) complete and, more surprisingly, coordination on the risk dominated strategy may become stochastically stable. Endogenizing the network for complementarity games would probably give a prominent role to (complete) bipartite graphs.

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Appendix 1: the log-linear evolutionary process

In this appendix, I briefly describe the log-linear evolutionary process and how it relates to the potential.[16] The myopic best-reply process associated with Γ_g is defined as follows.

At time $t = 0$, the system is in an initial state $x^0 = (x_1^0, ..., x_n^0)$.

At time $t+1$, an agent i is chosen at random with probability $1/n$ and reevaluates his choice.[17]

If $u_i(A, x_{-i}^t) > u_i(B, x_{-i}^t)$ then $x_i^{t+1} = A$

If $u_i(B, x_{-i}^t) > u_i(A, x_{-i}^t)$ then $x_i^{t+1} = B$

If $u_i(A, x_{-i}^t) = u_i(B, x_{-i}^t)$, i is indifferent between A and B. Two possible tie breaking rules are the inertial rule $x_i^{t+1} = x_i^t$ and the randomized rule $x_i^{t+1} = A$ with probability $1/2$ and B with probability $1/2$.

I denote by P this stochastic process. P is a myopic best reply process with stochastic order of moves. P is a Markov process on the finite set $\{A, B\}^n$. By construction, absorbing states of P are Nash equilibria of Γ_g. More precisely, absorbing states of P are weak equilibria of Γ_g under the inertial tie breaking rule and strong equilibria under the randomized one. Since Γ is a potential game, every best reply increases the potential. This implies that from any initial configuration, P converges to a Nash equilibrium of Γ_g with probability one. Using the terminology of Markov processes, the only recurrent classes of P are the absorbing states. In short, there is no cycle.

Perturbations of Markov processes have been used to investigate evolutionary selection of equilibria, e.g., Kandori, Mailath and Rob (1993), Young (1993). Following Blume (1993) and Young (1998), I introduce the **log-linear perturbation** of P, denoted by P^β where $\beta > 0$.

P^β is defined as P, except that if agent i is drawn, he chooses A with probability

[16] The reader is refered to chapter 3 of Young (1998) for a presentation of perturbed Markov processes and their use in evolutionary game theory. In this appendix, I only give a sketch of the argument.

[17] Exact probabilities do not influence stochastically stable states, if they are fixed and strictly positive for every player.

$e^{\beta u_i(A,x_{-i}^t)}/(e^{\beta u_i(A,x_{-i}^t)} + e^{\beta u_i(B,x_{-i}^t)})$ and B with probability $e^{\beta u_i(B,x_{-i}^t)}/(e^{\beta u_i(A,x_{-i}^t)} + e^{\beta u_i(B,x_{-i}^t)})$. P^β is another Markov process on $\{A,B\}^n$, approaching P (under the randomized tie breaking rule) as β gets large.

P^β has the nice property of being irreducible, i.e., there is positive probability of moving from any state to any other state in a finite number of periods. On the contrary, P in general is not irreducible, because of the multiplicity of Nash equilibria of Γ_g. An irreducible Markov process has a unique stationary distribution, i.e., a probability distribution over states invariant through the process. If μ^β is the unique stationary distribution of P^β, a state x is **stochastically stable** if $\lim_{\beta\to+\infty}\mu^\beta(x) > 0$. Roughly, stochastically stable states are equilibria of Γ stable under the log-linear perturbation.

The potential property greatly simplifies the evolutionary analysis. In fact, the potential allows one to obtain an exact expression of the stationary distribution μ^β of P^β.
$\forall\beta > 0, \forall x \in \{A,B\}^n, \mu^\beta(x) = e^{\beta\varphi(x)}/\sum_{y\in\{A,B\}^n} e^{\beta\varphi(y)}$ (see Young (1998), th. 6.1.). Therefore:

Proposition (Young, 1998, th. 6.1) *Stochastically stable states of Γ_g under the log-linear perturbation are maxima of the potential.*

Of course, alternative perturbations of P could lead to different stochastically stable states. To give a simple example of this, consider the following tree with 13 agents and a complementarity game such that $p = 1/2$..

	1	
2	3	4
5, 6, 7	8, 9, 10	11, 12, 13

1 is linked with 2, 3, and 4. 2 is linked with 1, 5, 6,and 7, and similarly for 3 and 4. Since trees are bipartite, maxima of the potential are

		A				B	
state 1:	B	B	B	and state 2:	A	A	A
	A, A, A	A, A, A	A, A, A		B, B, B	B, B, B	B, B, B

Now, under uniform trembles the following state is stochastically stable

		A	
state 3:	B	B	A
	A, A, A	A, A, A	B, B, B

The reason is that a single mutation of agent 4 is sufficient to go from state 3 to state 1 and vice versa. (This is similar to the fact that for coordination games and uniform trembles, the risk dominated equilibrium on the star is stochastically stable).

Appendix 2: complementarity vs coordination

In this appendix, I show how complementarity games played on graphs relate to coordination games played on graphs. Consider a complementarity game Γ and an graph of interaction g.

First, note that the potential of a vector of strategies x is the opposite of the potential of x for an associated coordination game $\tilde{\Gamma}$ played on the same graph:

$\varphi(\Gamma, g, x) = (a-d)n(AA)+(b-c)n(BB) = -[(d-a)n(AA)+(c-b)n(BB)] = -\varphi(\tilde{\Gamma}, g, x)$

with $\tilde{\Gamma} = \begin{bmatrix} d,d & b,a \\ a,b & c,c \end{bmatrix}$.

Therefore the attraction basin of the complementarity game Γ is the mirror image of the attraction basin of the coordination game $\tilde{\Gamma}$. Highest potential equilibria of $\tilde{\Gamma}$ are lowest potential states of Γ and vice versa. For coordination games, states of high potential are states with many links between similar agents. On the contrary, states of high potential for complementarity games are states with few links between similar agents, and many links between dissimilar agents.

Second, a priori being different from people with whom you are linked should be related to being similar to people with whom you are not linked. To see how this intuition translates on the potential, denote by g^c the complementary graph of g, i.e., $\forall i \neq j, g^c_{ij} = 1 - g_{ij}$, and h the complete graph, i.e., $\forall i \neq j, h_{ij} = 1$. In a matrix sense, $g + g^c = h$. Notice that the potential function can be rewritten in a matrix formulation (see section 4.5)

$$\varphi(\Gamma, g, x) = 1/2[(a-d)^t xgx + (b-c)^t(1-x)g(1-x)]$$

where x is the vector such that $x_i = 1$ if agent i plays A, $x_i = 0$ if agent i plays B, and 1 denotes the vector of ones. Therefore, the potential is a linear function of the graph, and

$$\varphi(\Gamma, g, x) = \varphi(\Gamma, h, x) - \varphi(\Gamma, g^c, x) = \varphi(\Gamma, h, x) + \varphi(\tilde{\Gamma}, g^c, x)$$

The potential for a complementarity game played on a graph is the sum between the potential for the same complementarity game played globally and the potential of the associated coordination game played on the complementary graph. Complementarity and coordination are dual notions, with respect to games and to graphs.

Appendix 3: a representation of $g_{4,2}$

The following picture describes an equilibrium of highest potential on $g_{4,2}$ when the relative payoff of playing Black is greater than 2 times the relative payoff of playing White. (All the other highest potential equilibria are isomorphic to this one). The proportion of Black agents is *3/4* on the complete subgraph in the center, but *5/12* for the overall graph.