Algebraic Reasoning: Developmental, Cognitive and Disciplinary Foundations for Instruction

Aims and Objectives of Algebraic Reasoning Conference

Daniel Berch (NICHD/NIH)

Although a burgeoning array of curricula for algebra and pre-algebra are being developed and marketed, comparatively few of these instructional approaches and materials are sufficiently grounded in rigorous research. Moreover, with some notable exceptions, pedagogical debates concerning the ability of primary, if not pre-K, children to acquire putative algebraically-relevant concepts (e.g., recognizing patterns) are generally uninformed by the most recent theoretical and empirical advances in our understanding of cognitive development in general or mathematical cognition in particular. Concomitantly, the vast majority of research to date in the domain of mathematical cognition has tended to be rather narrowly focused on basic numerical and arithmetic processing, with comparatively little effort to extend this type of work to the study of the development of algebraic reasoning skills. This paucity of knowledge is particularly vexing in light of the documented complexities associated with the transition from arithmetic to algebra, as well as the conflicting perspectives proffered by various factions regarding the appropriate sequencing of mathematical content in the elementary and middle school years. For example, whereas some have argued that a solid grounding in traditional arithmetic principles and skills is crucial for learning algebra, others contend that the "algebrafying" of arithmetic in the early elementary grades will subsequently ease the transition to pre-algebra in middle school and basic algebra in high school.

In an effort to shed some light on this often heated controversy, the co-sponsors thought it would be timely to organize a conference focused on the developmental, cognitive, and disciplinary (i.e., mathematical) foundations for instruction in algebra, with the following general aims: a) examine what we know and don't know about the requisite developmental and cognitive competencies for proficient pre-algebraic and algebraic reasoning, and how best to address the gaps in this knowledge base; and b) analyze what kinds of math problems should (or should not) be categorized as algebraic in content from the perspective of the field of mathematics. Conference participants will possess a wide range of expertise, drawing from such diverse fields as developmental psychology, educational psychology, cognitive neuroscience, math education research, and mathematics. Through formal presentations, as well as the portions of the meeting devoted to comments and questions and plenary discussions, we are hopeful that extant theoretical perspectives will be evaluated in light of relevant empirical data, and in addition, that suggestions for future research priorities will emerge. In our view, a firm knowledge base will be crucial for informing the development of effective instructional approaches to improve the acquisition of algebraic concepts and skills.

Meeting Summary

Wednesday, September 14, 2005

Welcoming Remarks and Meeting Logistics- Tom Loveless, Brookings Institution

Dr. Tom Loveless, Director of the Brown Center on Education Policy, welcomed participants to the proceedings, noting the encouraging presence of a heterogeneous group of policy-makers, mathematicians, and developmental psychologists.

Introductions- Presenters and Conference Co-sponsors

Introductions were made around the table.

Aims and Objectives- Daniel Berch, NICHD/NIH

Dr. Daniel Berch introduced the objective of the conference, namely to explore the gaps in our knowledge of the developmental and cognitive competencies required for becoming proficient in algebraic reasoning.

Understanding of Symbols at the Transition from Arithmetic to Algebra: the Equal Sign and Letters as Variables- Martha Alibali, University of Wisconsin

Dr. Martha Alibali presented her current work, focusing on pervasive symbols encountered on the student's road to algebra: the equal sign and letters used as variables. She noted that students need to be able to use these symbols fluently, as they are fundamental to algebraic understanding. By the time students reach the transition from arithmetic to algebra, they have been using the equal sign for several years, but they typically have not encountered letters in mathematical contexts. Dr. Alibali asserted that prior knowledge of symbols could be a stumbling block for students in transition, setting up expectations that are not appropriate for the learning of algebra. Dr. Alibali considered several questions in her research:

- How does understanding of the equal sign change across grade levels and how does it vary across task contexts?
- What meanings do students ascribe to letters used as variables, and how do these meanings change across grades?
- How does students' prior understanding of letters as abbreviations affect their interpretation of letters used as variables in algebra?

At the elementary and middle school levels, there are two main ways in which students view the equal sign. One is an "operational" view, in which the equal sign is treated as a signal to perform the operations given in the equation. According to this view, the equal sign "tells you to do something". This view allows students to succeed in some but not all contexts. The preferred view, from a pedagogical perspective, is a "relational" view, i.e., the equal sign represents a relation between two quantities, an equivalence or sameness.

In the 5th through 8th grades, many students possess an operational understanding of the equal sign. They say that the equal sign represents “the answer to the problem” or “what the problem adds up to”. Some students at these grade levels recognize the relational meaning of the equal sign, and say that it means “the same as”. A few students seem to recognize the relational meaning, but only in the context of the “greater than” and “less than” symbols (although one such student could not remember whether the symbol had 2 or 3 lines). Understanding increases across grade levels, but performance is weak across the middle grades. Only 45% of 8th graders in one Wisconsin sample offered the desired, relational interpretation. There is also variability in understanding the equal sign across task contexts. In 7th grade, for instance, students are much more likely to offer relational interpretations when they are asked about the symbol in the context of an equation with operations on both sides (e.g., $3 + 4 + 5 = ? + 5$) than in the context of an equation with only the answer blank on the right-hand side (e.g., $3 + 4 + 5 + 3 = ?$). In one study, most 7th graders (85%) offered a relational interpretation in some contexts. However, educators need to learn how to help students apply this more sophisticated understanding more broadly.

Extensive practice with arithmetic facts occurs throughout elementary school (e.g., $3 + 8 = ?$) and such experience may reinforce an operational view of the equal sign. Middle school students often miscode equations with operations on both sides (e.g., $4 + 8 + 3 = 4 + ?$), converting them to the more familiar form (e.g., $4 + 8 + 3 + 4 = ?$). Alibali hypothesizes that previous experience with equations in the typical format may promote the persistent operational view.

Does the operational view hinder students? Perhaps. Evidence for this comes from the mistakes students make on equivalent equations problems (e.g., Do the equations $3 + x = 15$ and $3 + x - 3 = 15 - 3$ have the same solutions?) Correct responses on such problems are more frequent among students who have a relational understanding of the equal sign than among students who do not. Some students must actually compute the equations to obtain the correct response, but others can recognize the equivalence without performing the computation. Students with a relational understanding of the equal sign are also more likely to correctly solve linear equations (e.g., $3x + 12 = 25$), even controlling for math ability.

Letters used as variables constitute another hurdle to algebraic understanding. What meanings do middle school students ascribe to letters used as variables? Past research has documented students’ as well as adults’ difficulties with variables. Students often treat variables as labels or abbreviations; e.g., d = dimes, rather than d = value of dimes. The abbreviation interpretation can be a hindrance for students at the transition to algebra. Students sometimes interpret variables as standing for multiple values, sometimes as standing for a single value, sometimes as standing for objects (e.g., n stands for newspapers), and sometimes in other ways (e.g., as a one digit number). In one 6th through 8th grade cohort in Wisconsin, approximately 50% of 6th graders held a multiple values interpretation, and there was improvement across grades.

Does prior knowledge or experience with letters influence students’ interpretations? Students encounter abbreviations throughout the curriculum, and textbooks frequently use the first letter of the object represented by the variable (e.g., n for newspapers). These experiences may reinforce the abbreviation interpretation. In one middle school sample, over 20% of students made the abbreviation misinterpretation when asked to interpret an algebraic expression in the context of a story problem.

Does having a multiple-values interpretation matter? When students were asked: “Which is larger, $3n$, or $3n + 6$?”, those who held a multiple-values understanding were more likely to answer correctly and to provide a correct justification than those who did not hold such an interpretation. Furthermore, the specific letters used as variables influenced the likelihood of proper interpretation. In a problem about cakes and brownies, when the letters *c* and *b* were used as variables, fewer students provided correct interpretations than when *x* and *y* or Greek letters (psi and phi) were used.

Alibali hypothesizes that students learn the meanings of symbols by deriving expectations from prior experience, and by making inferences based on implicit learning in the contexts in which symbols occur. Of course, opportunities for explicit learning about symbols are also important. However, it is not yet clear what types of learning experiences are most effective (e.g., direct instruction, bridging from prior knowledge, presenting contrasting cases). Patterns of performance suggest that students’ prior knowledge can sometimes be a stumbling block. Therefore, educators must be thoughtful when presenting symbols to students at the transition from arithmetic to algebra.

Discussion

Dr. Roger Howe commented that he was familiar with a book that explores the understanding of the equal sign with an unknown on one of the sides, and observed that the percentage of kids answering correctly actually decreased from 2nd to 6th grade. Dr. Alibali replied that kids have multiple interpretations and use them in different contexts, and excessive practice shows that kids can ignore unknowns; instead they activate their procedural knowledge about adding. Context activates relational and operational modes. Dr. John Anderson commented that there is another negative transfer phenomenon seen with algebra in relation to teaching some of the older programming languages (e.g., n to n +1). Dr. Alibali agreed that more knowledge could be worse in some contexts. Dr. James Milgram observed that in common usage in mathematics, there are many interpretations for the equal sign that are not mentioned. Natural language interferes with mathematics if not actively counteracted through instruction. Precise usage must be emphasized; Dr. Milgram further asserted that Dr. Alibali had presented sheer evidence of failure of instruction. Dr. Alibali agreed that implicit learning is going on, primarily.

Dr. Dave Carraher commented that getting into the interpretation of mathematics by students is important. The equal sign can be a cipher to most children; it’s almost as if kids don’t appreciate the transitive and symmetric properties of the sign. It is hard to separate development from the effects of teaching. However, it is possible to get kids to appreciate specific conventions associated with the sign. When dealing with letters, quantities and units of measure, this can be taught in a scalar way. It is important to make clear whether one is teaching the symbols in terms of referent transforms or exchange functions. When is it appropriate to introduce units of measure? Fourth graders can talk about this subject successfully, if properly taught. Dr. Alibali countered that the variable is tricky because of the varying contexts; the many different meanings constitute a challenge. Dr. Kenneth Koedinger expressed a fondness for the language learning metaphor, and added that the key developmental question is: does practice hurt acquisition of meaning? Earlier instruction of the relational meaning might make it easier. Dr. Alibali responded that some data suggests that students with stronger arithmetic skills are actually worse at the relational understanding of the equal sign. In studies of second language learning, critical period dynamics and entrenched knowledge of the first language may hinder acquisition of the second language. One may also think of the issue as a matter of entrenchment and not biological windows closing. It may thus be easier to teach (variables) at an earlier point. Asked to construct the ideal experiment to illustrate this difference, Dr. Alibali replied that there are different curricula that use the equal sign in different ways, therefore one might contrive to design a study in entrenched knowledge vs. non-entrenched and simply measure outcome. Dr. Loveless noted that textbooks have much less practice in them than in the 1970s and that there is little in terms of explicit instruction on the equal sign. He assumed that students and teachers know the meaning, but that this may be a teacher effect as well. Dr. Alibali commented that in equivalent equation problems, researchers have shown that teachers actually compute equations instead of recognizing equivalence. Dr. Howe shared an anecdote about a "Mathematics methods for teachers" course: in a ratio problem, more than 80% voted for the wrong form of the equation. Dr. Alibali was not surprised by the description; textbooks shape teaching. She has found wide variability of interpretations of the equal sign. Skills-based books emphasize operational constructions. Essentially, Dr. Alibali believed that experiences should matter, but at this point there is no real data for this assertion. Dr. Jerome Dancis commented that the learning vocabulary analogy explains a lot. When teaching seniors, he still saw the use of "d=dimes".

Dr. Richard Askey commented on the potential utility of studying textbooks from other countries, saying that the use of pictorial algebra seems to greatly influence the likelihood of successfully acquiring algebraic knowledge. In studies of the U.S. vs. Japan, in comparing solving vs. guess and check strategies (the latter more likely to be used by U.S. students), Japanese students used algebra more frequently. Computational skills do not appear to be the hindrance. Dr. Alibali agreed that it would be very interesting to see how symbols are used in foreign textbooks. Dr. Susan Chipman asserted that it matters what one means by knowledge: rote and superficial vs. deep understanding of arithmetic. Those with a deep understanding of arithmetic would find algebra trivial. She refuted the common conception of children learning new languages more easily than adults; in light of some study results, one might argue that adults are better at the task. Dr. Alibali agreed that students who have a good number sense probably have an easier time with symbols. Dr. Ralph Raimi commented on the prior internalization of deplorable conventions. Elementary teaching gives incomplete arithmetic instructions. The equals sign doesn't say "do anything" until you provide some instructions in English. Dr. Alibali noted that the nature of the arithmetic knowledge structure is partially the issue, as it influences how students think about arithmetic.

Lessons from Early Algebra Research- David Carraher, TERC

Dr. David Carraher presented his research perspective as a teacher of young students. He viewed the teaching of early algebra in the context of early mathematics and approaches to it; as an area of teacher education and curriculum development; and as a research domain. Research about students was his primary concern: problems and how children conceptualize them. He presented the results of the second of two longitudinal studies in 3rd to 5th grade students (26 children in 2 classrooms). Items from MCAS (a skills test in Massachusetts), NAEP items, and some other internal assessment practices comprised the assessment criteria. When testing experimental students who had had some additional mathematics instruction during the experimental period, 35% of the experimental group vs. 8% of the control group scored 2/3 correct or better. Experimental students did better in algebraic reasoning. These are encouraging but not dramatic results. Early Algebra is not the same as Algebra Early. Examples of some ambiguous problems were presented boxes of candy with unknown quantities of candy inside them; the problem is about all the possible stories. One third-grader treated this as an empirical problem and shakes the box to determine the amount of candy. Another third grade student produced iconic drawings showing some variations of understanding of n+3, a good bridge to literal representation of an unknown. Logical constraints of the problem are generally recognized at this stage as well. When the activity is transformed to all the possibilities, children make the shift from empirical to affirmative.

Eighteen months into the experimental program being described, a wallet problem was introduced: Mike has $8 in his hand, the rest of his money is in his wallet. Robin has 3 times as much money in her wallet. In the experimental group, in the middle of 4th grade, a student could describe this problem as Nx3=3N. Another student presented a tabular representation of the problem. Strategies for scoring, from weaker to stronger, were single value : indeterminate drawing or statement : multiple values : general expression. There was a dramatic shift in conceptualization over 18 months' time. An interesting research question is what happens when the child makes the transition from describing the situation to producing the written syntactical form. Should one start from syntax? This is hard to do in some cases. Dr. Carraher felt it was more valuable to drive from situation to form. He presented an observed difficulty with filling out values in tables as it related to a word story example. Symbolic headers allowed the children to omit the story this case, showing that symbols can mediate thought and simplify it. Graphs can play roles in conceptualization as well. The exploration of a parameter space, i.e. monotonically filling up the wallet, can be viewed as a dependent vs. independent variable phenomenon. Graphs and algebraic notation enrich student thinking and create a "meta-story". An interesting question is: how do students move between these interpretations of mathematical lessons: algebraic script, tables, graphs, and certain structures of natural language? One can assume that mathematical ideas are expressed in different representations- their interchangeability is not immediately obvious to kids. Students need to piece together this information in different representational systems. The contribution of psychological research is to clarify how students interpret mathematics; these are not misconceptions but models in elaboration. It is important to redefine mathematical teaching with a concerted effort in research.

Discussion

Dr. Joan Moss noted similarities between her work and that of Dr. Carraher, in that there are some children who are able to understand patterns and functions, but as soon as another representation is introduced, they get lost. One must think about this as further sequences of instruction evolve. Dr. Carraher responded that educators must design classes. Teachers are doing curriculum development whether they like it or not. He cited having extended discussions with mathematicians on when to introduce tables and graphs, for example, and hoped to achieve this by the end of 5th grade instruction, with algebraic input. He described wrestling with issues of order of introduction, among many other issues. Dr. Koedinger asserted that Dr. Carraher had presented evidence that young kids can learn algebra. Is it a question of should they or can they learn algebra? If they are learning algebra (at a given grade level), are they losing out on something else? Dr.

Carraher replied that the students are already experiencing loss in the form of impoverished mathematics teaching. While the issue of subject displacement is an important one, educators must view more comprehensive mathematics teaching as a deepening of the existing curriculum rather than a replacement. Dr. Koedinger asked: to what extent are pictorial representations emergent? Dr. Carraher replied that in order to bring proof to the fore, instructors varied parameters, and concluded that these pictorial representations are spontaneous. The logic of simplifying impressions takes many forms. Dr. Carraher wanted to push the concept toward the notion of equal operations on both sides of the equation. Dr. Koedinger asked if teaching this pictorial representation benefited students. Dr. Carraher felt that one must carefully consider how things are arranged in space. At a certain point the student must adopt the technique and rely on it. He didn't want to introduce the concept of equivalence as a blind rule. Dr. Loveless aired his concerns about using manipulatives where the goal is to move to numbers as a general concept. Manipulatives can get in the way of learning. He asked Dr. Carraher to comment on the use of concrete objects and manipulatives. Dr. Carraher agreed that he shared Dr. Loveless' concern, but related an anecdote of successfully using a clothesline as number line, and to expand the concept to illustrate infinity. He felt that kids could distinguish between the symbol and the number for which it stands, and could handle the discussion about idealized objects. These results, however, are anecdotal and not formally presented. Dr. Greg Solomon cited some similar work on rational numbers. Dr. Raimi asked if Dr. Carraher had presented his 5th grade students with a representation (graph or column) and asked them to make up a story in response. Dr. Carraher applied in the affirmative, adding that the children actually generated more stories in response to sparser notation; this richness is what one brings to mathematics.

The Potential of Geometric Sequences to Foster Young Students' Ability to Generalize in Mathematics: A Report from Second and Fourth Grade Research Classrooms in Diverse Urban Settings- Joan Moss, Ontario Institute for Studies in Education

Dr. Joan Moss presented results of her work with second and fourth graders in using geometric sequences in generalizing mathematics concepts. She regarded her work as being at the pilot stage and cited previous work in collaboration with Robbie Case on rational numbers and neo-Piagetian concepts. It has been hypothesized that patterns may constitute a practical way to help children generalize, and by extension, to develop an understanding of the dependent relations among quantities that underlie mathematical functions. In addition, young children appear to have a natural interest in patterns, and older children similarly find patterns intriguing. Manipulation of patterns also spans ability levels, rendering the practice suitable for most curricula.

Dr. Moss presented a number of video sequences illustrating classroom use of patterns, and observed that children typically come to this activity with a limited idea of patterns as repeating sequences. In general, children can successfully extend patterns but have difficulty describing and generalizing the pattern far down the sequence.

Dr. Moss reported being struck by how geometric growing sequences need to be taught to young children- understanding this concept is evidently not a natural ability. It is difficult for children to go from patterns to useful rules. At the beginning of the pattern instruction curriculum, young children have a tendency to use the scalar and recursive rule. Whole object reasoning leads to the incorrect use of a proportional strategy. These kinds of problems don't go very far. In early encounters, Dr. Moss reported that once students selected a rule for pattern, they persisted in claiming the rule even when encountering flaws in their hypothesis. It is believed that the cause for this persistence is a disconnection between actions, results, and expression of rules. Another cause is the lack of perceptual agility, or the ability to abandon rules that do not work. Dr. Moss's pilot research involves having children engage with geometric and numeric patterns in ways that can forge connections between different pattern representations; illuminate mathematical structures underlying patterns; and provide natural solutions for mathematical problem solving, discussion, reflection, and knowledge building.

The theoretical framework of this pilot research rests on focusing on geometric and numeric patterns independently. Initial results were presented for a second-grade study conducted in three populations. The lesson sequence consisted of geometric patterns and position cards, geometric composite functions, sidewalk building, a function machine concept, and a walkathon. Data collection was performed via videotape, field notes, pre- and post-test interviews, and administration of Case's number knowledge test during the pre-intervention phase to determine developmental level. Results revealed that while the control group was higher on the number knowledge level at the beginning of the study, the experimental group improved by nearly 2 standard deviations in explaining patterns functionally and explicitly. Qualitative results demonstrated that children were able to understand features of multiplication and its applications through functional reasoning, and also the concept of zero. However, the children were not always able to use functions in a variety of settings. They displayed difficulties with continuous movement between representations, and difficulty with composite function challenge. The children in second grade had not yet been exposed to multiplication, but were still able to determine functions in function machine setting, complete "tables and chair" problem settings, and to construct arrays for multiplication. It was further observed that children would start to talk about zero when building patterns, and that the concept emerged independently in separate groups.

Discussion

Dr. Dancis commented that the students seemed to have derived numbers from the geometric sequences to look for patterns and that it almost seemed that the patterns were separate from the geometry. Dr. Moss replied that the intent was to start geometrically growing structures and to determine number of elements. The function machine used in the pilot study presented another view of this process- input and output cards were used, and children were challenged to find the numbers that connected them. It was hoped that the children who could use the geometric model could move from recursive to truly functional ideation. Dr. Dancis felt it important to argue from the appropriate source, whether extrapolating from geometry or science. By itself, seems that the geometric approach is inadequate; one needs to justify the generalization. Dr. Moss agreed and reiterated that the intent was to get children to generalize from the model. Dr. Koedinger noted that in getting children to state the rule numerically or symbolically, the children seemed to be able to get the right result for the far generalization case, but could not seem to articulate the rule and had difficulty symbolizing it. Dr. Moss reported that none of the control groups ever expressed rules, and similarly observed that the idea of generalizing was elusive. Dr. Madge Goldman asked: what is the correlation with the verbal skills test? The problem could originate with verbal skill deficiencies, not math understanding. Dr. Moss replied that in the grade 4 level of the studied population, 40% of the children were not English-speaking.

What Makes Algebra Hard for Learners?- Kenneth Koedinger, Carnegie Mellon University

Dr. Kenneth Koedinger's work focuses specifically on algebraic problem solving in terms of computer simulations of how students think, and the development of *in silico* Cognitive Tutors. A key question is: Should algebra be taught in earlier grades? If the answer is *No*, is it because they are not ready, or if the answer is *Yes*, is it because fluency needs to be gained over time? The answer may vary based on algebraic content and knowledge. The issue here is what is hard for students, and when. Dr. Koedinger leaned toward introducing algebra in earlier grades, but called for more evidence to uphold this idea. He presented some data on which types of problems were more difficult for beginning algebra students: story problems, word problems, or equations. For first graders, the story problem is harder than the matched equation. Past studies had found that story problems were difficult, words were hard to comprehend, and a need for translation hindered the problem-solving effort. In spite of these general claims, no one had done the matched comparison in algebra.
Recent findings indicate that story problems are actually easier for beginning algebra students, and these studies have been replicated. Dr. Koedinger felt that poor algebra performance is probably a consequence of insufficient instruction; an inability to solve equations is rooted in shallow knowledge in equation solving. He further noted that math teachers and researchers are prone to "expert blind spot" whereby their own intuitions about what is hard (story problems) cloud the reality of student difficulties. He also described an eye-tracking study of students that provides further evidence for the idea that learning math formalisms is much like learning a foreign language.

Problem-solving
Problem-solving ability (situation and problem models) has been studied in terms of a comprehension phase and the solution phase. A verbal advantage, that is, beginning algebra problems stated in words are easier to understand than those stated in symbols, is seen to be a component of this ability. Error profiles have shown that equation comprehension is hard and that the difficulty occurs at the comprehension stage. Dr. Koedinger presented results from a sample of students from various locations throughout the U.S. and the world, including Russia. There are definitely issues of curriculum that are relevant. Why are equations harder? By junior high, students have better reading comprehension of English words, while at the same time the demands of comprehending symbolic expression increases going into algebra. A verbal advantage is also seen for college students for simpler problems. But for higher complexity problems, story problems are indeed harder. The benefits of symbolic abstraction to aid problem solving become clear for higher complexity problems. However, the observation remains: comprehending and producing algebraic symbols is hard. Should algebraic symbolism begin to be taught in middle school? Yes, like learning any foreign language, learning the symbolic language of algebra takes time.

Bridging instruction
Dr. Koedinger presented some instructional experiments of algebra learning performed in the context of a computer-based Cognitive Tutor program. In the control condition, the Cognitive Tutor used a textbook-style sequencing of problem-solving prompts going from the abstract to the concrete (write a formula for a story problem and then use it to solve cases). In the experimental condition, the Cognitive Tutor used an "inductive

support” sequencing of the problem-solving prompts going from concrete to abstract (solve cases and then write a formula that generalizes across them). Students learned more (as measure by pre-to-post test improvement) from the inductive support instruction illustrating how effective instruction can bridge from students prior concrete conceptions to learning new formal abstractions. Other instructional paradigms have used a “bridging context” to build on prior informal knowledge of function (earnings from a walkathon) toward abstract functional representations in tables, graphs and equations. Evaluation in grades 6, 8 and 10 have shown greater student learning from this approach than a traditional algebra course. In summary, students can reason about unknowns before formal algebraic instruction, the language of symbolic algebra is difficult to learn, early exposure to use of language reduces the formal-informal gap, bridging instruction from informal to formal is effective, but more research is definitely necessary.

Discussion

Dr. W. Stephen Wilson commented that Dr. Koedinger might want to try his approach on people who have had a real algebra course (such as one based on the Forester textbook). Something is very wrong when people can’t solve simple linear equations after eight months of an algebra course. He suggested more algebra practice as a solution, and to separate true algebra problems from mere arithmetic. Dr. Howe commented that if all we want to do is teach students to solve a linear equation, algebra instruction is not necessary. Dr. Richard Askey asked if problems had been presented with simpler numbers, separating the issue from computational problems. Dr. Koedinger replied that decimal problems were not difficult- students were used to dealing with money, and that while whole-number performance is better when dealing with setting up word or story problems, the same pattern persists. Dr. Askey mentioned a useful book (published in 1910) that problems without the use of numbers. Dr. Cathy Seeley asserted that the goal of (modern) instruction was to use concrete examples as a bridge to motivate symbolic learning, using the informal understanding that students possess, and connecting this understanding to the desired mathematical outcomes. Most people would say algebra should be taught early. Dr. Koedinger believed early instruction was beneficial, but called for more supportive evidence. Dr. Seeley asserted that the teaching of arithmetic is actually the basis of algebraic understanding. Dr. Raimi questioned the utility of pattern recognition exercises in 2nd grade. Dr. Koedinger felt that such exercises at that grade level provided good preparation, and helped students to learn from examples and induce generalization. Dr. Raimi questioned whether early algebraic instruction was indeed cost-effective.

Algebraic Reasoning: David Geary

Dr. David Geary presented the day’s ideas in the context of evolutionary psychology, and asked the audience to look at mathematics in a broader perspective, all the way back to the Pleistocene era (from 1.8M to 11,000 years ago). Mathematics is a human activity of very recent cultural origin. What is the human mind prepared to learn about it? Why doesn’t mathematics come easily? How mathematics should be taught depends critically on what it is treated as; how it is taught will interact with what is learned easily from an evolutionary perspective and what is not learned easily; the latter is predicted to include most cognitive and intellectual domains of recent cultural origin. There is a built-in transfer from evolve domains and as related to prior learning. When mathematics may be taught should be considered in the terms of necessary background knowledge, and in terms of cognition and development.

Is the human mind inherently prepared to learn algebra? Probably not. Dr. Geary argued that primary human abilities center around folk psychology (e.g., language, social competencies), folk biology (e.g., knowledge of other species), and folk physics (e.g., ability to navigate). There also appears to be a very rudimentary and potentially evolved understanding of number, counting, and simple addition and subtraction. Neuroimaging research reveals these quantitative abilities may be subserved by the parietal cortex, that is, many of the same systems that support folk physics.

In any event, evolved abilities emerge through interaction between inherent biases and child-initiated activities such as play and social discourse. Secondary abilities, in contrast, do not emerge as effortlessly through children's self-initiated activities, but rather are built through folk systems via effortful attentional control, working memory, and content knowledge provided in schools. Algebra is one of these secondary abilities that will only emerge with schooling. In fact, teachers, textbooks, and school are needed to provide the organization children to attention and learning such that algebraic concepts and procedures can be built from more primary folk knowledge.

When viewed in terms of cultural innovation, advances in mathematics, the sciences, technology and so forth create a gap between the store of cultural knowledge and built in folk knowledge. Schools emerge in societies in which these gaps need to be closed. One function of schools is to allow children to acquire the competencies that close the gap between recently emerged cultural and intellectual advances and folk knowledge.

The implications are important for understanding what children are biases to learn, what they would prefer to learn, and why these biases and preferences may be at odds with the demands of school-based learning. Children are innately curious and motivated to actively engage in activities that will develop folk knowledge (e.g., learning about social relationships), which can conflict with the need and motivation to engage in the academic activities needed to learn algebra and other forms of evolutionarily novel knowledge. Moreover, inborn human cognitive systems and child-initiated activities may not be sufficient for the easy acquisition of secondary abilities, such as reading, writing, and mathematics. The need for formal instruction is a direct function of the "remoteness" of the secondary ability to the supporting primary systems. Algebraic word problems, for instance, can be made easier than formal mathematical representation of the same problem, because word problems can be presented in terms of a natural social exchange. The algebraic expression is much more "remote" than the story problem.

Humans are "naturally" equipped with numerosity (the ability to assess small quantities without counting), an ordinal sense, a counting sense, and the ability to detect increases and decreases in small quantities (arithmetic). It is necessary to ask: what are the primary competencies that help and hinder algebraic learning, both procedural and conceptual competencies? How do humans naturally think about patterns? What is the role of language vs. symbols?

The more remote the secondary competencies from the primary, the more instructional material must be organized by teachers and mathematicians. The period of instruction for acquiring secondary competencies will be longer and more preliminary preparation will be needed. For many children the material will be subjectively difficulty and inherently uninteresting. Motivation to learn the material comes from adults' understanding of what is needed for successful social living and preparation for technical fields, and not the inherent interests of children. In this vain, mathematicians and educators must determine how to conceptualize mathematics, as a set of applied skills to aid in day-to-day living or as a formal academic discipline. The forms of mathematics that is taught to reach these goals will be very different.

Dr. Dancis argued that algebra could be presented in a manner that is of basic interest to students. Dr. Geary asked if this sort of presentation was better for long-term outcomes, and felt that the answer to this question was unknown. He presented extensive samplings from first through sixth grade Chinese arithmetic and algebra-practice workbooks, and pointed out the well organized, extensive practice for speed and accuracy, and that pre-algebra was integrated into already practiced arithmetic. In a Grade 1 example, there were columns of addition with many commutative pairs. Double-digit addition and subtraction were added to the curriculum in the second semester of Grade 1. By the second semester of Grade 2, workbooks included mixed operations, and assumed knowledge of order of operations. In addition, practice was timed. In the first semester of Grade 3, the workbooks introduced missing values and the concept of *x* as a variable. By Grade 3, 2nd semester, students were drilled in multiplication with variables. By Grade 4, decimals and operations were introduced, and by Grade 6, complex expressions with multiple operations, fractions mixed with decimals, and percentage equivalencies appeared. Dr. Loveless commented that he had visited China in 1985 and noted that student scores were publicly displayed; students strove to be the fastest and the best. Dr. Geary asserted that Americans were much better at this discipline in the early 20th century.

Dr. Geary concluded his presentation by suggesting that algebra has gone beyond folk knowledge and that the need for children to use algebra is in fact quite recent in origin. The question should be the core content: is algebra to be treated as a discipline or an applied skill? Some studies suggest that prior secondary learning interferes with symbolic understanding of algebra. Algebra is remote from primary knowledge and involves layering and integrating various types of secondary symbols. Children's natural use of symbols is more about social communication and simple visual representations; use of symbols in other ways may be need to be explicit. When considering attempts to fuse learning preferences, educators must consider how primary representations help and hinder this learning. Is the Chinese method as sufficient as other approaches to teaching mathematics and algebra? Are there other approaches we have not considered?

Do attempts to make learning democratic and authentic, help or hinder algebraic learning? Dr. Geary thought these attempts may hinder learning and operated from a basic assumption that children are not motivated to learn algebra, and that there is now a great potential to influence secondary ability acquisition and reduce interference. At present, researchers need to determine what cognitive systems are used in the acquisition of algebra, and how the demands of algebra differ from the evolved function of these systems. One cannot make assumptions about teaching algebra too early because there is insufficient data. Chinese workbooks suggest that American children can be introduced to pre-algebra in a systematic way earlier than is the current case. Dr. Dancis suggested that both teachers and textbooks were capable of motivating children. Dr. Geary replied that one could hook them by addressing their primary interests, but that strategy in and of itself is not likely to be insufficient.

Discussion

Dr. Koedinger asked whether the Chinese method is known to be effective, adding that Singapore students do well but they also have bridging representations. Dr. Geary pointed to the superior performance of the Chinese relative to other countries, but acknowledged that assumptions could not be made. Dr. Alibali expressed interest in the idea that readiness might not matter. What do we mean by cognitive developmental readiness, and what are the stepping stones? Dr. Geary cited the ability to focus and attentional control as prerequisites to learning algebra. It is still an empirical question. It is know that the brain continues to mature into adulthood. It's not a matter of spending more time, it's spending time in better ways, such as in the discipline of practice and drilling. Dr. Wilson asserted that ten minutes of practice per day is enough. Dr. Moss cited motivation and secondary abilities; kids can do many things that are not "folk" abilities. Being familiar with the prerequisite is what helps children to launch into difficult tasks, and many children find these secondary tasks compelling. Dr. Geary argued that everyone needed to know these things, not just motivated academicians. Engagement is necessary but not totally sufficient. What is the ideal mix? Dr. Anderson speculated about the advent of formal schooling; acquisition in apprenticeship mode vs. formal schooling is not simply primary vs. secondary learning. Dr. Geary replied that kids would be likely to imitate hunting, but not the learning of algebra. Things that were not necessary for the population at large a century ago are now necessary- everyone must read, for instance. Dr. Chipman observed that there are people who like mathematics, and that most (other) people have absolutely no idea what mathematics is good for beyond the 8th grade level. Which occupations require what levels of math? Dr. Dancis offered his notion of the modern stumbling blocks to learning algebra: poor textbooks and incompetent teachers, and misguided ideas about learning and democracy. Better textbooks, better teachers and better ideas about democracy (i.e. ranking and competition) could foster learning algebra. Algebra should be the next civil right; it is the crucial door to the middle class. Dr. Anderson asked: how much of this is credentialing vs. use of algebra? Dr. Askey proffered an anecdote describing how WWII navigators experience with calculus helped their proficiency with trigonometry, pointing up the need to determine which skills are necessary to understanding certain levels of math. Dr. Seeley felt that most provocative statement was questioning the teaching of algebra as "interesting". Dr. Geary felt that some people will find algebra interesting, but that this fact does not allow one to assume that what motivates a small segment of the population will motivate everyone else. Dr. Seeley cited this observation as a basis of the effort to make algebra more "relevant" and interesting. Dr. Geary asked: as you change the curriculum to make it more interesting, is it departing from formal mathematics ideas?

Plenary Session

Dr. Berch began the discussion by citing data showing the U.S. as scoring high in interest in math but scoring poorly in its practice, and that therefore it must be acknowledged that all effort need not be "fun". Dr. Carraher raised the issue of expecting students to express their own ways of thinking. It is easy to fall into polarities and it is wrong to assume that even highly motivated children are capable of generating systems that took centuries to develop (they are not going to reinvent mathematics). Very often teachers elicit ideas from students as opportunities to introduce conventions. One must look at children's natural representations but must correct them in terms of conventions. Practice is necessary; mastery does not preclude discovery. The process of working in workbooks offers opportunities for insights. There needs to be a balance. Dr. Howe offered the example of his daughter, proficient in math, and who is known for being good at spreadsheets, which she attributed to being good at algebra. By extension, he felt that in dealing with the dichotomy between primary vs. secondary abilities, it's part of the art of the teacher to make subjects real and immediate. In the case of the Chinese workbooks, it seems that exposure to patterns was an operative in developing skills; it is important to structure math instruction in this way. Dr. Susan Sclafani remarked that the issue of teacher quality has been skirted; teachers need to show the beauty and value of these solutions. The implications for teacher training at all levels must be discussed. Current teachers do not start at a level of deep mathematical understanding. Dr. Madge Goldman commented that kids seem to be uninformed and incurious. This outcome is totally foreign to what a school should be. There is something far more seriously wrong with the educational system, spanning all subjects. Dr. Wilson observed that most mathematician-educators have expertise in dealing with freshman and sophomores; teachers must never refuse to teach an interested 10-year-old the appropriate math. Everyone should be given the opportunity to pursue higher math. Dr. Koedinger acknowledged that the consensus was that the situation might well be dire. What do we recommend, and to whom? Dr. Wilson added that further research was not necessary to determine the appropriate content of the math curriculum.

Thursday, September 15

Discussion/Wrap-up of previous day

Dr. Loveless began by outlining the economic view of the value of algebra, and acknowledged that this view has always been cloudy. The establishment and expansion of formal schools are rooted in the separation of the workplace from the family. In historical terms, the seat of education had been the family and church. When the church and state separated, the U.S. developed a secular institution of learning. As the country moved from agriculture to industry, parents left home, thus creating the need for teachers. This was a massive change. In developing countries today, urban areas still tend to have the most efficient mass schooling; this is much less true in the rural areas. Dr. Dancis asserted that the bottom line in learning algebra is whether or not children know algebra at the end of 9^{th} grade. They need to know arithmetic going in, and they need a competent teacher. In addition, they require fluency in arithmetic word problems and understanding of the equal sign. Geometric patterns are important, but it is not crucial students learn that type of analysis before taking algebra. Dr. Dancis posed the question: to what extent does the inclusion of algebraic comments in K-8 prepare students for algebra in 9^{th} grade? Dr. Koedinger replied that it takes extensive practice to learn a skill. The Chinese curriculum showed that more years spent in preparation allowed achievement of competence. Dr. Dancis agreed, adding that familiarity with arithmetic takes years to build; the time used for algebra preparation is often time taken away from arithmetic. Dr. Carraher believed that arithmetic skills could be strengthened by the teaching of some algebraic concepts.

Fitting functions earlier into the curriculum show initial encouraging results. Acquiring fluency in conventional notation may be valuable by the time students enter 9th grade. Educators may be underestimating the role of practice. Dr. Milgram offered a mathematical perspective of the situation: Arithmetic only occurs in the K-1 curriculum. Things change dramatically when base 10 number place values are introduced; the basic structure of algebra is being introduced. If introduced appropriately, algebra should be integrated early, as in other countries that manage to teach algebra successfully. Dr. Seeley agreed. Dr. Solomon noted that many researchers in cognitive development would agree with Dr. Geary's greater point that young children, perhaps with innate support, are able to reason in sophisticated ways about aspects of the psychological, biological, and physical worlds, as well as about mathematics. By contrast, most professionals outside of the field of cognitive development tend to think of cognitive development in terms of an orthodox Piagetian framework; that is, they believe that until adolescence, children are unable to reason in an abstract causal fashion. Decades of studies have undermined this strict Piagetian view, though it still heavily influences elementary curricula. The point to make is that the difficulties children have in learning algebra may have less to do with their general reasoning abilities, and more to do with how specific concepts are taught. Dr. Solomon then noted that Dr. Geary's argument for algebra instruction that heavily emphasizes drilling did not follow from his characterization of cognitive development and is a separate position for which no empirical support was presented. Dr. Hyman Bass noted that some intuitive introduction to algebra is important, along with literal notation and variables, as it is also tied to language learning. Children can begin to invent ways of expressing infinite quantification; this gives birth to the need for algebraic notation, a natural way to introduce this in the early grades.

The Algebraic Brain- John Anderson, Carnegie Mellon University

Dr. John Anderson presented observational studies on the neurological underpinnings of algebraic ability, and began by refuting the notion that algebra was a matter of primary vs. secondary skills. None of the primary skills, as previously discussed, appear to be uniquely human abilities; all can all be shown in primates. Language appears to be uniquely human, however, as well as the ability to acquire an unbounded array of secondary skills, and algebra is one of these. Brain imaging studies appear to support the notion of primary mathematical skills, but not of secondary skills. The brain develops throughout adolescence and adulthood. Is there an age at which algebra teaching becomes appropriate? The parietal and prefrontal regions of the cortex are undergoing extensive change during adolescence. There appears to be no connection in the response (to algebraic learning) between adults and developing children in this respect.

Dr. Anderson presented fMRI (functional MRI) studies in a model of how children learn, representing varying results from three ROLE-funded studies by Yulin Quin; the first was a base study of how children learn how to solve equations, the second a study of how adults solve equations, and the third a study of how adults learn an artificial algebraic system. The first experiment involved children from 11-14 years old who had mastered middle school math but had never solved an equation, and were just about to enter Algebra I. On Day 0, instruction and coaching on how to solve 0-2 step equations was successfully accomplished. On Days 1-5, the children engaged in computer-based practice on these 0-2 step equations. (By comparison, the much-discussed Chinese algebra workbooks contained one-step equations). Students responded to questions via data gloves that measured responses and response time, and were imaged in an fMRI scanner on Day 1 and Day 5. An ACT-R computerized learning model was used to illustrate the order of functions in problem solving. In the ACT-R module, memory and brain function steps are processed through a production system. The ACT-R system was given verbal instruction similar to that given to the test subjects. The model solved problems differently (less complexly) as it became more proficient. Both the model and the children performed faster on Day 5 than on Day 1. Collapsing of steps and faster retrieval of arithmetic facts was seen over the learning period. Through the learning process over the 6 days of the experiment, the time required to solve a 2-step equation on Day 5 was roughly the time taken to solve a one-step equation on Day 1

Two parameters were estimated, latency scale for retrieval and the time to visually parse. There is a great deal of theoretical complexity for a simple set of numbers. Predictions based on the ACT-R parameters showed the cortical analogs of function: fusiform gyrus, motor, prefrontal, basal ganglia, and the anterior cingulate gyrus, etc. The motor cortex was correlated with manual response, the parietal cortex with imaginal functions, the prefrontal cortex was activated during retrieval, and the anterior cingulate gyrus was associated with goals, etc. The 11-14 year olds were as accurate as college students at this task.

The responses seen in the brain represent hemodynamic demand on the region being tracked. In the case of the motor cortex, response peaked after 4 to 5 seconds of the finger pressing the answer button. Lag time did not change, but response time did. The prefrontal/retrieval region showed a much stronger response as operations increased. The magnitude of the response decreased over time because as the subject practices, there is less information to retrieve. The patterns are very similar in both children and adults. The anterior cingulate gyrus/goal region showed almost no change in response with practice, indicating that the same amount of demand was being placed on the system. The same responses were also seen in adults both practicing algebra, and learning the artificial system. The parietal/imaginal response was similar to the anterior cingulate gyrus (in children), but in this area one also saw a strong learning result- the subjects were starting to skip steps. This same response was also seen in adults solving algebra equations.

The regions studied in these experiments are general information processing regions that achieve arbitrary secondary competencies. Verbal insight experiments reveal activity in the same regions. The parietal region activates just before the insight, and the anterior cingulate gyrus reacts afterwards. The anterior cingulate gyrus is a “hot” area in science; a recent study shows this area to be associated with mediating response conflict. In terms of hemispheric specialization, traditionally one sees mostly left-brain involvement in many learning tasks, but studies show right brain involvement when using geometry proofs.

Discussion

Dr. Loveless asked: what will the child’s fMRI profile look like without prior instruction? Dr. Anderson replied that studies are just beginning for this condition: the short answer is that learning time is extended. There is a break point with practice, where too many facts are lost and practice becomes a waste of time. Dr. Alibali asked how the behavior of the model was plotted. Dr. Anderson provided an explanation, adding that the hemodynamic response was actually an increased oxygenation of the blood. Dr. Alibali asked if more potential conflict (as seen in the anterior cingulate gyrus) might show higher activation. Is activity in this area expected to decrease with practice? Dr. Anderson replied that he was in the process of looking at this task in adults who have problems solving algebra. Dr. Solomon asked: What are the next steps for finding effects at an individual level, as in remedial activities? Dr. Anderson replied that the practical payoff in measuring the effects of practice is creating a synthetic student. The artificial construct can be carried forward in instructional design by developing instructional intervention. Researchers are working toward observing tutorial interactions in the fMRI scanner and identifying those interventions that are working. This is one of the possible promises of this research. Dr. Suzanne ? raised a current dilemma: do we need to remediate first, or have them learn as they go? One piece of data suggests just putting struggling students in the class and have them learn as they go. Are there certain levels of practice that yield faster retrieval, or other ways of looking at skills that enhance algebra learning? Dr. Anderson acknowledged that a substantial research effort would be required to provide such answers. Dr. Bass asked: do these methods work in problems where there is a fork in the decision tree? Dr. Anderson replied that he had performed these studies with models, not humans. The methods do work with models. Dr. Chipman related information about an automated skills for cognitive readiness project, wherein researchers were looking at changes in brain activity as skills become automated; executive areas of the brain gradually decrease in activity as a skill is acquired. Eventually EEG measurements could be used to help pinpoint relevant activities. There is interest from the military in such studies, especially as automated skills are resistant to stress, fear, and alcohol. These studies may be relevant to deciding how much drill and practice one requires in these areas.

Panel Presentation: Analysis of NAEP Items Classified Under the Algebra and Functions Content Strand

Panelists: Roger Howe, Hyman Bass, James Milgram

The panel discussion was devoted to reviewing items from the NAEP, the "national report card". Algebra and functions test items were examined. The review centered around four main questions:

- Are the items mathematically sound?
- Are the items algebra?
- Does the mastery of the items predict success in algebra?
- Are important topics not being covered in the NAEP inventory?

Dr. Bass began the discussion, explaining that to be mathematically sound, the items must be clear, age-appropriate, and free of non-purposeful ambiguity. He found that the ensemble of items represented the mathematical domain, and that individual items satisfied the first two criteria. For the third criterion, most items sufficed. The exception was an item that involved a piece of distracting context (sea level item). Another example, concerning the weight of puppy over a series of months, when carried out to its logical conclusion, shows that the puppy loses weight and eventually disappears. In terms of validity (the psychometric question), does success in solving these problems predict a successful algebra student? One problem in particular, Grade 4 #14, appears to be a test of the knowledge of even numbers. A student might perform the progression and identify the right answer without knowledge of the property of even numbers, therefore one cannot predict the psychometric value from this particular question. What is algebra?

Traditional thought holds that algebra is the theory of polynomial or algebraic equations. Current thought centers around functions and patterns, modeling data, and mainly linear (sometimes quadratic) and exponential functions. The learning emphasis in traditional curriculum was in instinctively solving linear and polynomial equations and achieving symbolic fluency. Dr. Bass suggested that the current curriculum is a weak caricature of real modeling functions; an algebra student needs to sort out messy situations. Knowing arithmetic (computational fluency) does not assure algebraic fluency. Patterning exercises occupy too much space at the expense of more traditional items. Some topics that should be included in the NAEP assessments are: simplifying radical expressions and finding a quadratic polynomial $f(t)$ such that $f(0) = 1$, $f(1) = 0$, and $f(-1)=1$.

Dr. Howe agreed with many of Dr. Bass' conclusions. The essential aspects of algebra are working with variables and general expressions, and algebraic structure (rules of arithmetic- a neglected aspect of algebra that accounts for much current failure). Arithmetic must be taught in a way to bring out an appreciation of how its rules influence manipulation of variables. This is neglected in NAEP problems. The attention to patterns and the notion that "mathematics is the science of patterns" seems questionable. The range of patterns children are exposed to is quite restricted; patterns appear to be just another type of rote, stylized performance. With respect to the NAEP pattern problems at the 4th grade level, of the 9 presented, 3 were properly formulated, and 6 were implicit patterns. These latter 6 problems were not deemed appropriate for large-scale tests. One shouldn't expect kids to guess in mathematics, which constitutes a pedagogical cheapening or neglect of patterns. Dr. Howe was also struck by the relative lack of depth of the NAEP items. They were mostly 1- or 2-step problems, which did not involve some of the more difficult aspects of arithmetic. In the Grade 8 items, only two moderately difficult problems were presented: solving 2 systems of linear equations (one given graphically, to solve by intersection) and one horses/chickens/legs problem (a low-performance problem). There were few problems on order of operations, and the items were also arithmetically simple, with no problems on distributive laws. The test does not constitute a thorough probing of an Algebra 1 course.

Dr. Milgram primarily addressed the NAEP Grade 8, observing that many of the problems were mathematically incorrect; and that virtually all the problems were at a low level relative to U.S. math standards, and at an even lower level when compared to foreign problems. Of 41 items, 8 were incorrect, and 1 was meaningless.
Of 22 Grade 4 items, 4 were incorrect, 4 were vocabulary items, and 1 problem could be considered challenging. In the Grade 5 items, Dr. Milgram termed a dotted pattern problem as miserable; there was no continuation rule, and no discernible correct answer. A brief debate erupted over parsimony, with some participants responding to Dr. Milgram's criticism of pattern problems. Dr. Milgram claimed that mathematics does not care about parsimony.

Dr. Milgram displayed algebra problems from other countries as contrast (Russian Grade 4 = U.S. Grade 5). The definition of terms was provided at an age-appropriate level in Russia; these definitions are not offered until 9th or 10th grade in the U.S. The typical Grade 4 Russian problems were more difficult than the Grade 8 NAEP. Dr. Koedinger observed that bioarithmetic could substitute for algebra given an infinite cognitive capacity, but added that the U.S. education system cannot ignore what is going on in foreign countries and their comparatively higher levels of expectation. U.S. expectations are not even close, and there is no indication that foreseeable future holds any hope. The cost of this neglect is increasingly enormous. Dr. Loveless offered a caution, in that the reviewed problems were not necessarily representative of the test.

Panel Discussion

Dr. Dancis asked: if a student scores well on the Grade 8 NAEP, will he or she perform well in Grade 9 (in the U.S. curriculum)? Dr. Bass replied that if the student essentially answers all questions correctly, the answer is yes. Dr. Milgram agreed that the same student might have a good conceptual basis, but perhaps not the fluency in notation and operations. Dr. Howe also agreed, noting that there were a couple of problems that combine arithmetic difficulty with some algebraic ideas (usually the most difficult), but confined his remarks to the notion of preparation. Dr. Bass commented that it was really a discussion of the curriculum. It will take a long time to bring U.S. students up to speed with foreign competency. It will be necessary to put in place a long-term agenda to move the curriculum in that direction. Dr. Koedinger felt that it was possible to empirically study performance in the boundary between algebra and arithmetic. It is known what leads to increases in cognitive load; e.g., when an unknown is referenced more than once (eliminating arithmetic strategies). Dr. Seeley echoed other cautious remarks in interpreting the released NAEP problems. A new framework was put in place in 2000 that ramped up the difficulty of the items. She expected that national tests would raise the bar by changing the definition of proficiencies as students increase their understanding of difficult problems. There is also a difference between test problems and classroom problems; this should be a context for NAEP developers. As more rigor and content is introduced, expectations are that grade-level mathematics courses would become more high-level. Increases in performance on the NAEP Grade 4 and Grade 8 items have been seen- this is encouraging. Dr. Seeley agreed that a long-term agenda is necessary, and that national tests should raise the bar on the test every year.

Dr. Askey remarked that preparedness for Grade 9 is not the problem. He cited a multi-step Grade 4 problem, asserting that many programs do not have students adding more than 2 numbers together. Performance drops dramatically when adding 4 numbers on the NAEP test. Teacher knowledge is also an issue; teachers must be able to perform in order to teach. Dr. Milgram noted that by law, educators cannot ramp up the difficulty of these problems dramatically. NAEP is constrained by existing standards (until 2007). He also expressed concern about the high level of errors in the NAEP items. Awareness of this latter problem has been raised, but in the meantime we do not know what the high error rate means. Dr. Pat O’Connell Ross remarked that NCLB stipulates that states develop requirements, leading to great variability. There must be a more immediate way of raising expectations and more capacity to change expectations and match the curricula. State assessments do not match with NAEP. Dr. Elizabeth Albro, Education Research Analyst at the Institute of Education Sciences, noted that Arlington County Public Schools have recently instituted a county-wide goal to have 100% of students completing algebra by the end of the 8th grade. Dr. Milgram reiterated the opinion that capacity is lacking at the teacher level. Dr. Chipman noted that NAEP was designed to assess how states are doing, not the individual student. It is a different assessment than that of the state. The process of item selection is not based on established theory. NAEP should be timing answers, for instance. Dr. Sharif Shakrani felt that the panel had assessed the released items well, but added that NAEP has changed since that time. The new assessments are more relevant to grade level. NAEP is intended to reflect what is being taught, but it is also meant to lead. Over time, algebra content in 8th grade has been rising and increasing in complexity. Dr.

Shakrani alluded to commentary at the state level over how hard the NAEP is compared to state assessments. He completely disagreed with Dr. Milgram's enumeration of incorrect items. The NAEP items had been thoroughly vetted. Dr. Askey added that pattern problems would not be present on future NAEP tests. Dr. Scalfani noted that states are loath to make changes unless pushed. A new NAEP can help push change at the state assessment level. The new NAEP framework can have a major role in this campaign. The business and higher education communities are deeply concerned and can also bring influence to bear. It is imperative to teach mathematics every year through the end of high school. Dr. Carraher appreciated the contributions of the panel and Dr. Milgram's comments on patterns, but wanted to draw out the issue of conjecture in mathematical reasoning; he felt that it is not all deduction, and that critics were excluding the possibility of insight in mathematics. Mathematics proceeds just like other sciences, from particular to general. Is there any way to work with patterns short of giving the general rule to the student? Dr. Milgram asserted that the type of analysis Dr. Carraher was citing was in fact pre-mathematics; it is done in the classroom looking at implicit hidden rules. One can't do this on a large-scale test. Guessing is destructive to problem solving and high-end reasoning.

Dr. Bass raised the question of how to reasonably reconcile patterning and mathematics. Multiple-choice is not amenable to Dr. Carraher's criteria for pattern prediction. He agreed that there should be a place for speculation; it is a real part of mathematics. Dr. Howe commented that there were serious issues about what can be done in a high stakes exam in terms of cognitive load. One could invent a good problem, but it could be gamed, as teachers teach to the test. The fairness demand trumps the desire to have challenging questions. Since NAEP is not a high stakes exam, it might be a good place to have high cognitive load problems. Dr. Raimi noted that in New York state, standards are not matched by the exams. Exams have been dumbed down. He urged that the NAEP include difficult questions. Dr. Loveless interjected that the NAEP is high stakes for the incumbent administration; the incumbent does not want to see scores fall. Dr. Bass remarked that teachers need to know much more than multi-step problems; they must also know the typical student shortcomings.

Summary and Wrap-up by Meeting Co-sponsors

Dr. Berch summarized the proceedings by inviting subsequent steps. From the perspective of NICHD, the next step would be to decide what research to fund in the developmental and cognitive disciplines. There are many gaps in this knowledge, and the evidence base is comparatively thin. Much ongoing work is neither programmatic nor longitudinal. A good number of questions posed during the conference were empirical. Educators don't want to be asking these same questions in 5 years with little additional data having been accumulated. Dr. Cordova thanked the speakers, and invited research proposals that target systematic and rigorous research on algebra. Dr. Solomon pointed out one opportunity in the National Science Foundation's ROLE program. NSF is currently planning a budget and looking to reshape the program; a small competition is coming up that could accommodate some of the questions that had been raised during the conference. Dr. Loveless felt some important themes included more research on the usefulness of patterns, and the possibility of "horseracing" curricula. Motivation is also crucial; educators need to worry about those who take the same algebra course 4 times. Can we motivate poor students, and if so, what are the tools to employ? The teacher's time is limited and therefore one must think about the most efficient use of the teacher's lesson plan. A participant made a plea to research agencies: they need to get out into the field and help schools frame their questions in terms of empirical research. Schools need to be helped in formulating their testing. Researchers must get out and educate the educators. Dr. Wilson thanked the organizers for convening the meeting. Dr. Bass concluded by remarking that the preparation of teachers is very important, and that one must empirically study how to translate deep knowledge of mathematics into the effective teaching of algebra.

NATIONAL INSTITUTE OF CHILD HEALTH AND HUMAN DEVELOPMENT
INSTITUTE OF EDUCATION SCIENCES, U.S. DEPT. OF EDUCATION
NATIONAL SCIENCE FOUNDATION
BROOKINGS INSTITUTION

Algebraic Reasoning: Developmental, Cognitive, and Disciplinary Foundations for Instruction

Brookings Institution, Falk Auditorium
1775 Massachusetts, NW
Washington, DC

September 14-15, 2005

AGENDA

Wednesday, September 14, 2005

A.M. 8:30 BREAKFAST

9:00 Welcoming Remarks and Meeting Logistics
Tom Loveless, Brookings Institution

9:15 Introductions

9:30 Aims and Objectives of the Meeting
Daniel Berch, NICHD/NIH

9:45 Understanding of Symbols at the Transition from Arithmetic to Algebra: The Equal Sign and Letters as Variables
Martha Alibali, University of Wisconsin

10:15 *Comments and questions*

10:30 BREAK

10:45 Lessons From Early Algebra Research
David Carraher, TERC

11:15 *Comments and questions*

11:30 LUNCH

P.M. 1:00 The Potential of Geometric Sequences to Foster Young Students' Ability to Generalize in Mathematics: A Report from Second and Fourth Grade Research Classrooms in Diverse Urban Settings
Joan Moss, Ontario Institute for Studies in Education

1:30 *Comments and questions*

1:45 What Makes Algebra Hard for Learners?
Kenneth Koedinger, Carnegie Mellon University

2:15 *Comments and questions*

2:30 BREAK

2:45 Discussant
David Geary, University of Missouri

3:30 *Comments and questions*

3:45 Plenary Discussion

4:30 ADJOURN

7:00 *Dinner*

<u>Thursday, September 15, 2005</u>

8:30 BREAKFAST

9:00 The Algebraic Brain
John Anderson, Carnegie Mellon University

9:30 *Comments and questions*

9:45 Panel Presentation: Analysis of NAEP Items Classified Under the Algebra and Functions Content Strand
Chair: Tom Loveless
Panel Members: **James Milgram, Roger Howe, Hyman Bass**

11:00 BREAK

11:15 *Comments and questions*

11:45 Summary and Wrap-up by Meeting Co-Sponsors
Daniel Berch, Diana Cordova, Tom Loveless, Gregg Solomon

12:15 ADJOURN