What Makes Algebra Hard for Learners?

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Conference question: Should algebra be taught in early grades?

- Differing views & rationales
  - No, because students are not ready
  - Yes, because students need to develop concepts & fluency over years

- Key issue here
  - Is the answer the same for all algebraic content & knowledge?
  - What is hard for students, when?

Overview

- Comparing matched verbal story & symbolic equations
- Analysis of strategies & errors
- Teach algebra in middle school?

- Bridging instruction studies

Effect of Problem Representation on Beginning Algebra Problem Solving

Which problem type is most difficult for Algebra students?

Story Problem

As a waiter, Ted gets $6 per hour. One night he made $66 in tips and earned a total of $81.90. How many hours did Ted work?

Word Problem

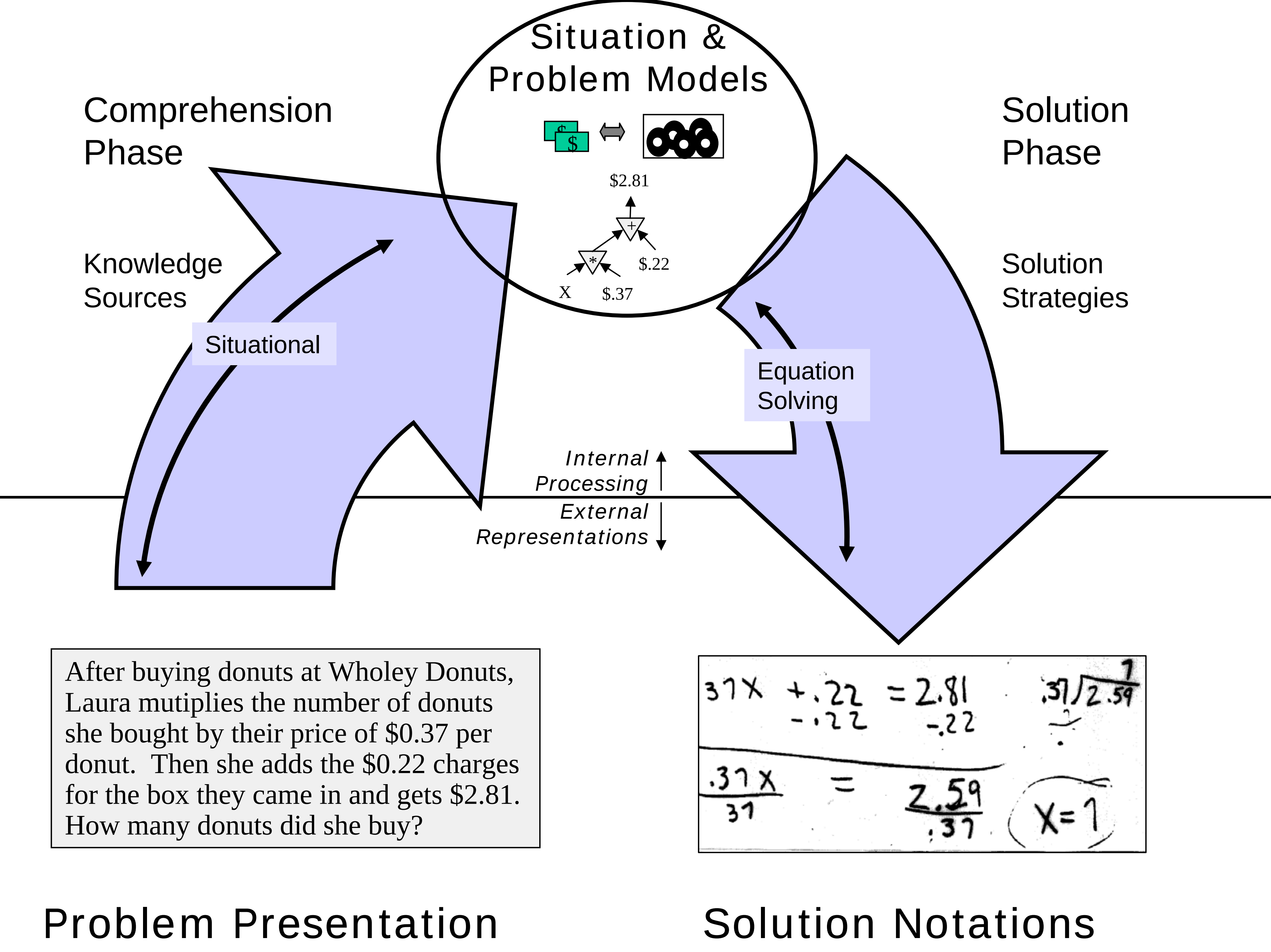
Starting with some number, if I multiply it by 6 and then add 66, I get 81.90. What number did I start with?

Equation

$x * 6 + 66 = 81.90$

Existing data & views: Story problems are harder

- First graders: Story problems are harder than matched equations (Cummins et al., 1988)
  - Generalization: "as students advance to more sophisticated domains, they continue to find word problems in those domains more difficult to solve than problems presented in symbolic format (e.g., algebraic equations)"
- General practice in algebra teaching
  - Educators: story is harder (Nathan & Koedinger, 2000)
  - Equations first in textbooks (Nathan et al., 2002)
- Comprehension & solution phase rationales
  - Hard to comprehend words
  - Need to translate
- *Yet, no one had done the matched comparison in algebra*



Situation &
Problem Models
$2.81
+
*
$.22
X
$.37
Comprehension
Phase
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Solution
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After buying donuts at Wholey Donuts, Laura mutiplies the number of donuts she bought by their price of $0.37 per donut. Then she adds the $0.22 charges for the box they came in and gets $2.81. How many donuts did she buy?
37X + .22 = 2.81
-.22 -.22
.37X = 2.59
X=7
Problem Presentation
Solution Notations

Hell's Library.

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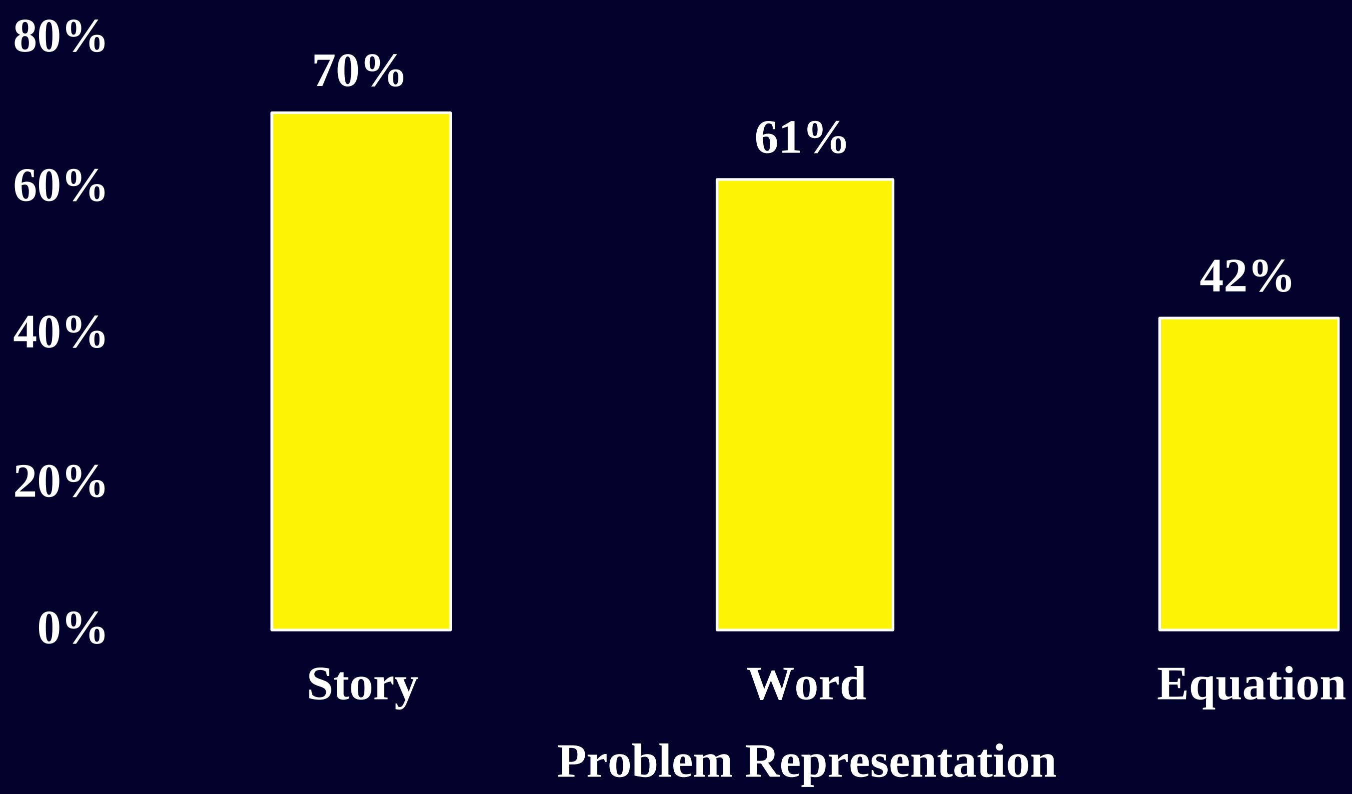
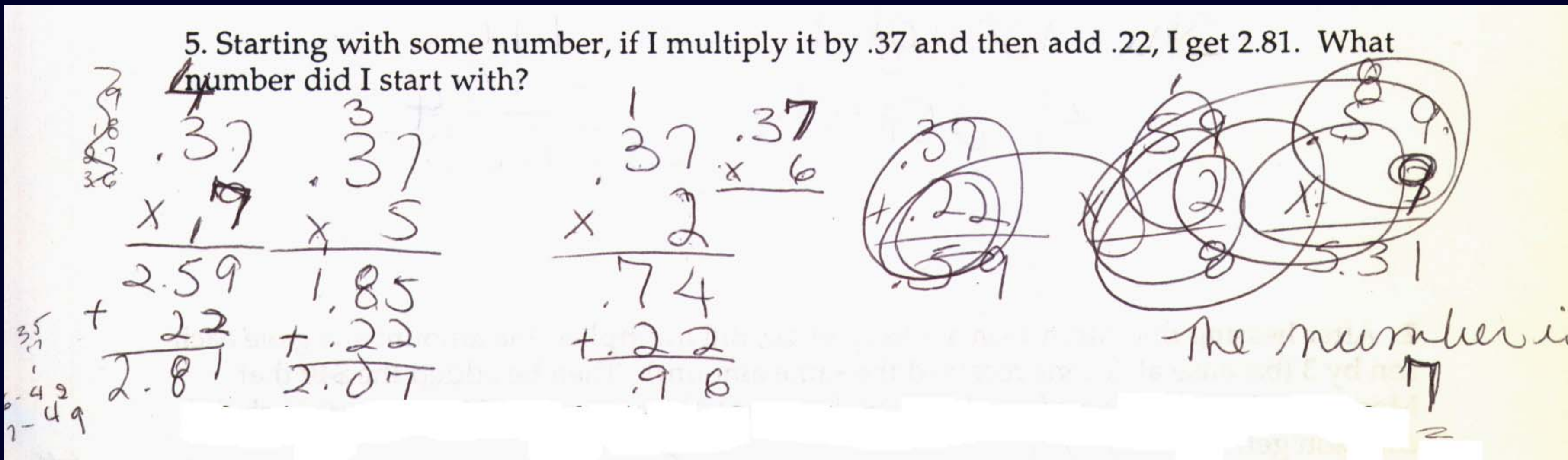
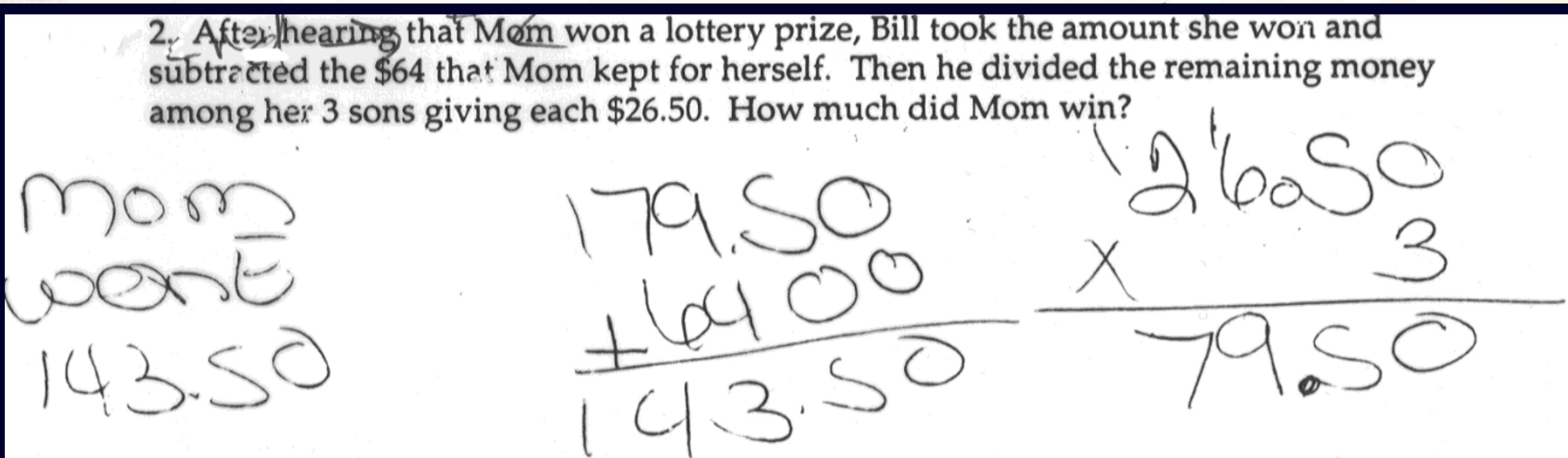
Algebra Student Results: Story Problems are Easier!



Koedinger, K.R. & Nathan, M.J. (2004). The real story behind story problems: Effects of representations on quantitative reasoning. In *International Journal of the Learning Sciences*.

Informal Strategies





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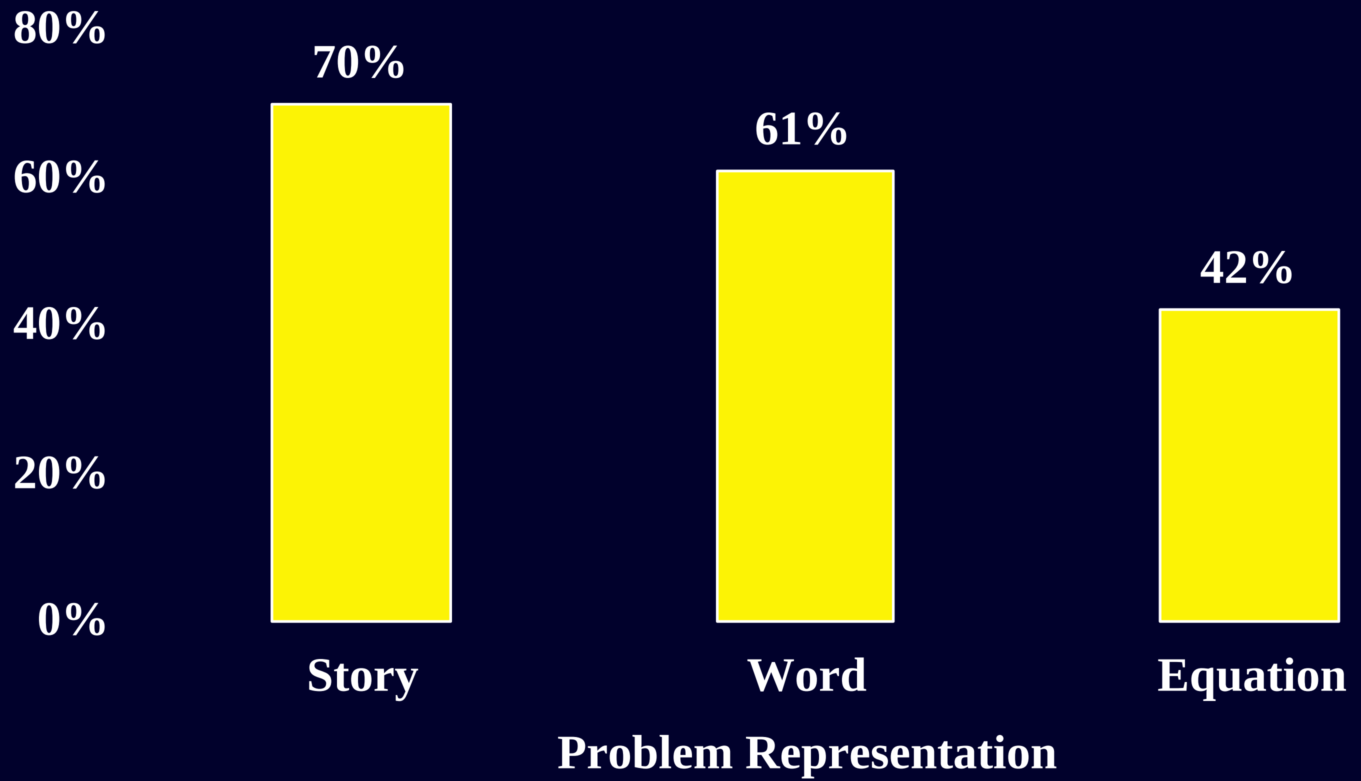

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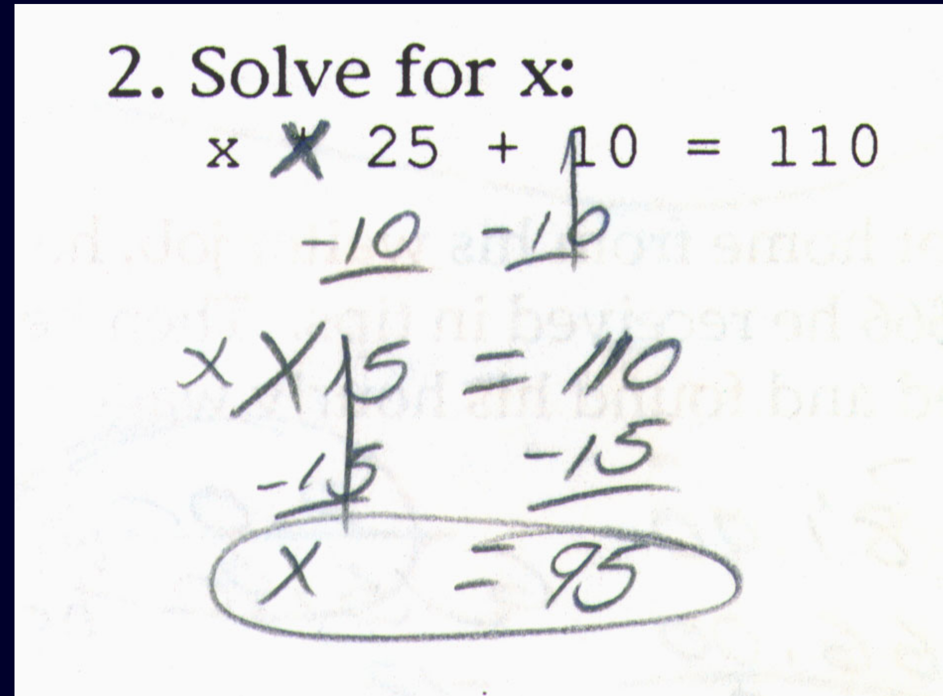
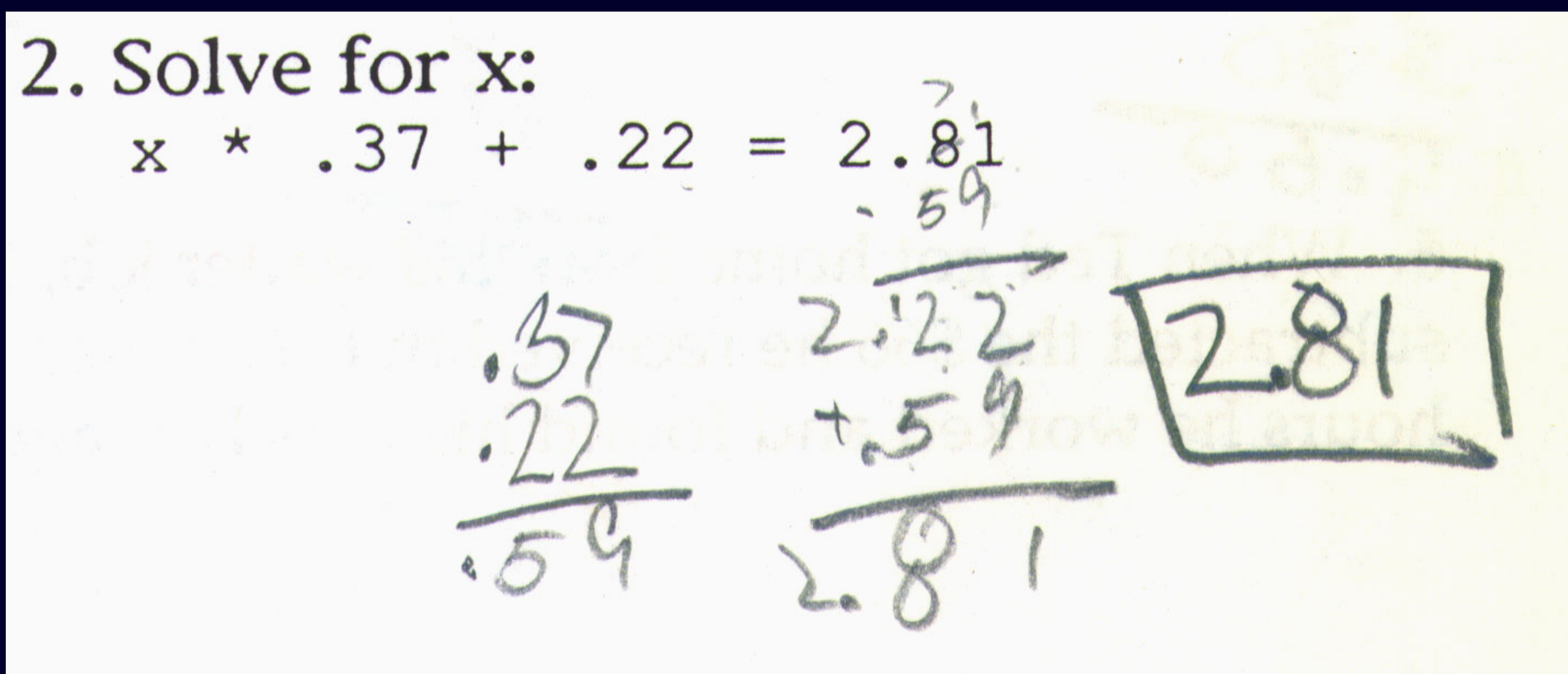
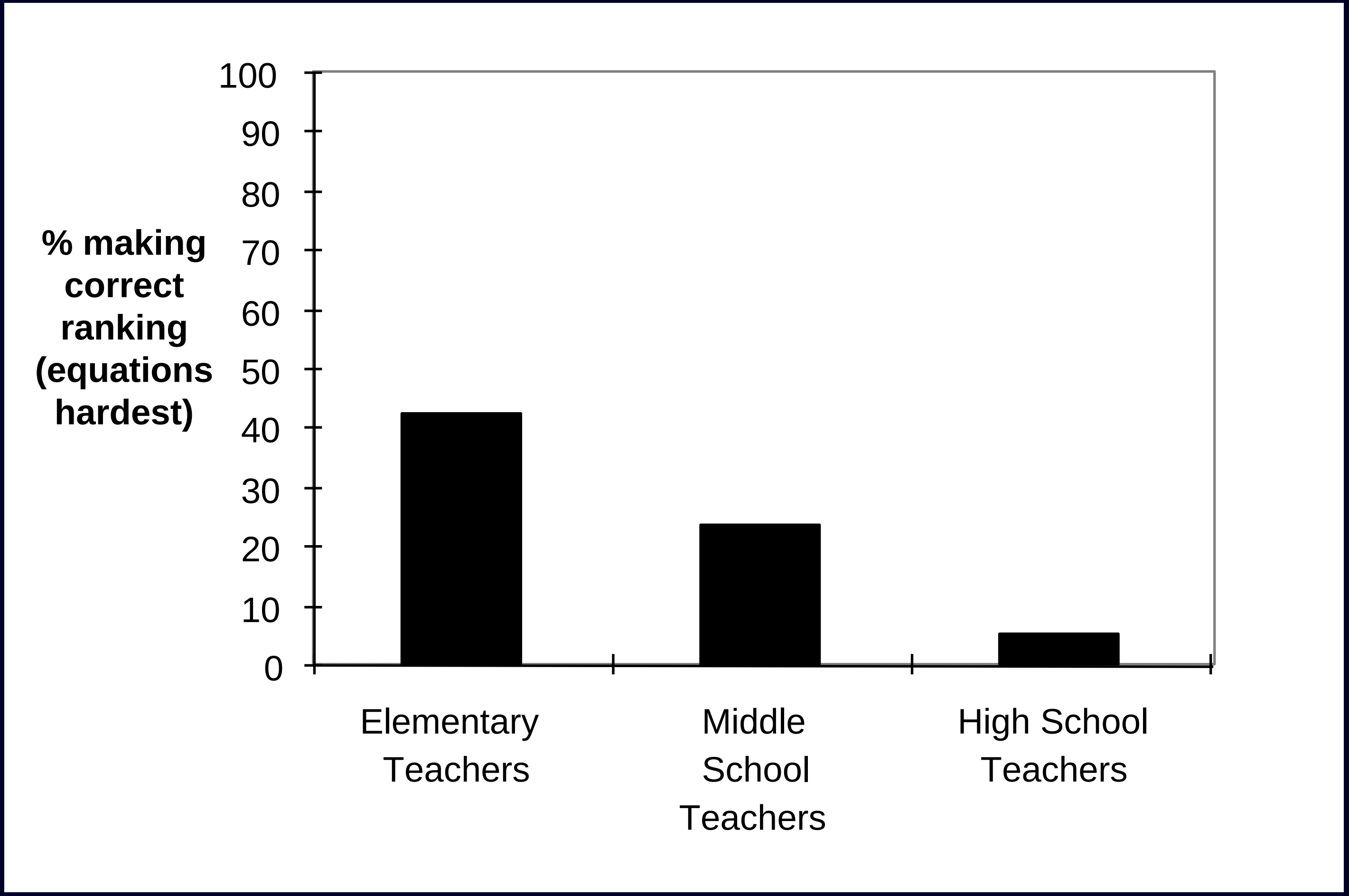
“2nd language” of algebra: shallow knowledge in equation solving





Expert Blind Spot

Algebra teachers worst at recognizing algebra student difficulties

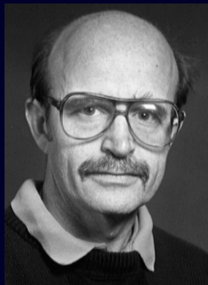


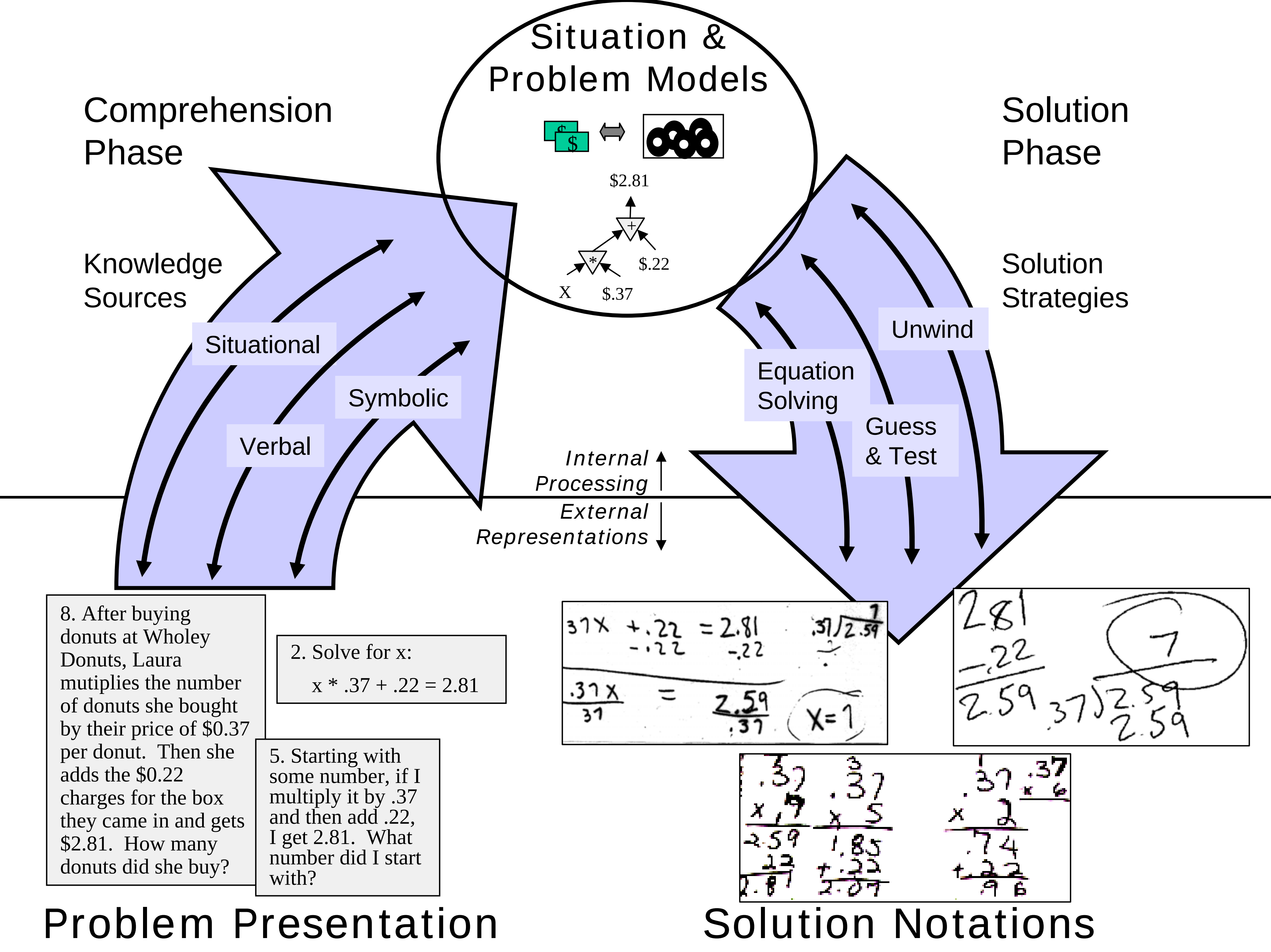
Nathan, M. J. & Koedinger, K.R. (2000). Teachers' and researchers' beliefs of early algebra development. *Journal of Mathematics Education Research*, *31* (2), 168-190.

Eye Tracking Studies:

Math formalisms are like learning a foreign language

QuickTime™ and a Video decompressor are needed to see this picture.





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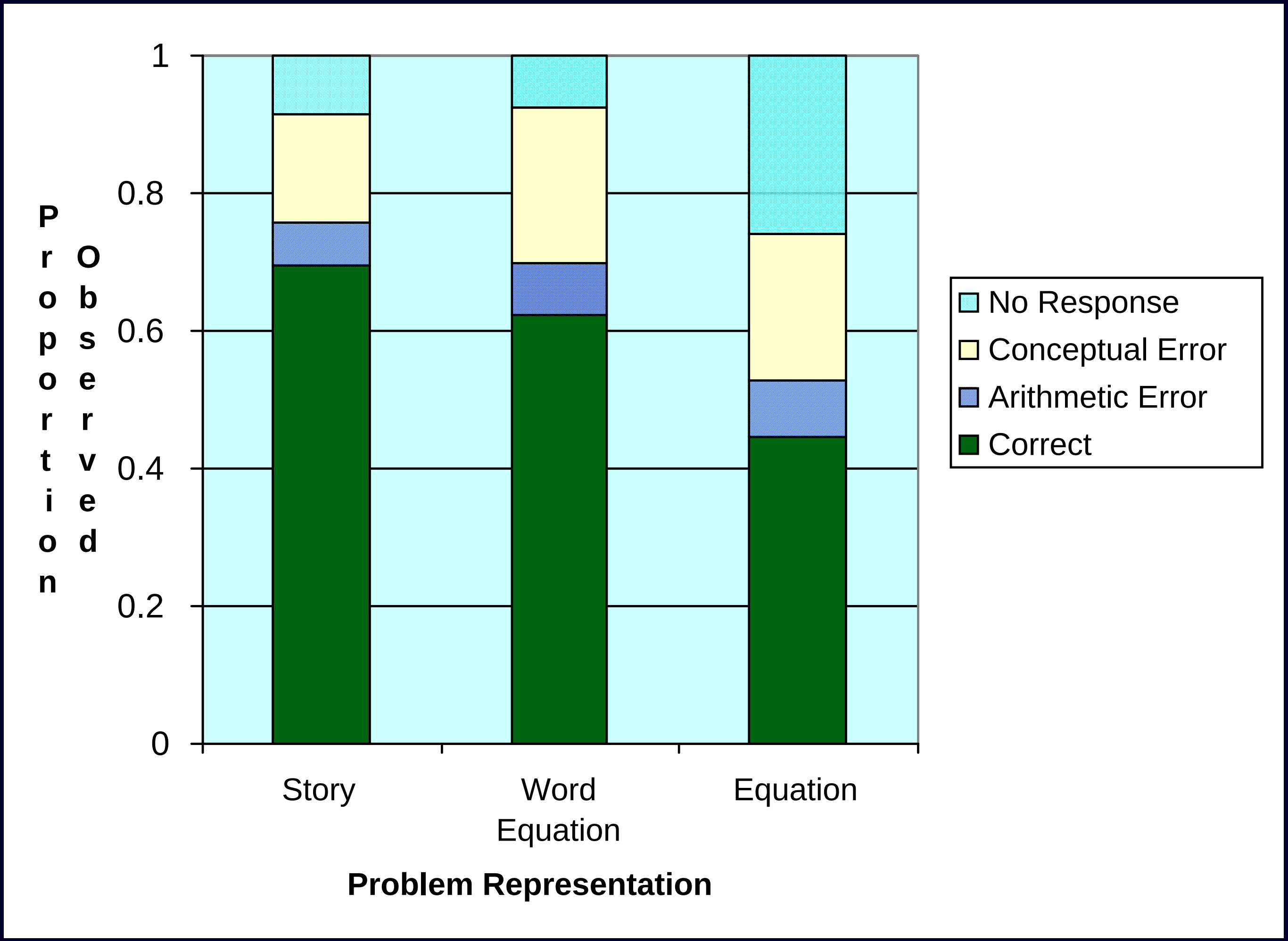
2. Solve for x:
x * .37 + .22 = 2.81
5. Starting with some number, if I multiply it by .37 and then add .22, I get 2.81. What number did I start with?
Problem Presentation
Solution Notations

Verbal advantage

- Multiple replications of this “verbal advantage”
  - Different student populations in a variety of contexts

- Strategy & error analysis
  - Ss have informal algebra problem solving knowledge prior to acquisition of symbolic equation solving skills
  - Ss key difficulties in learning algebra symbolism are essentially "language acquisition" issues.

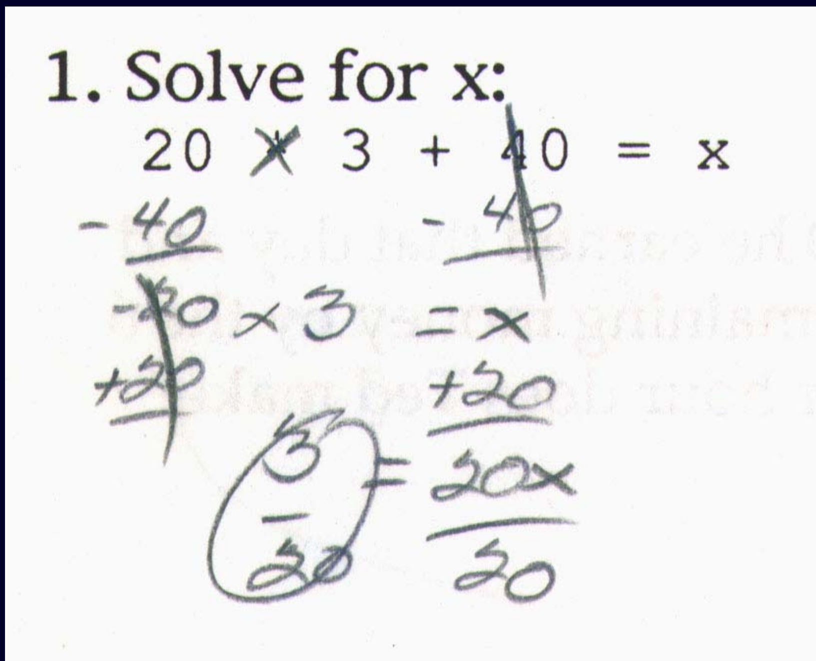
Error Analysis

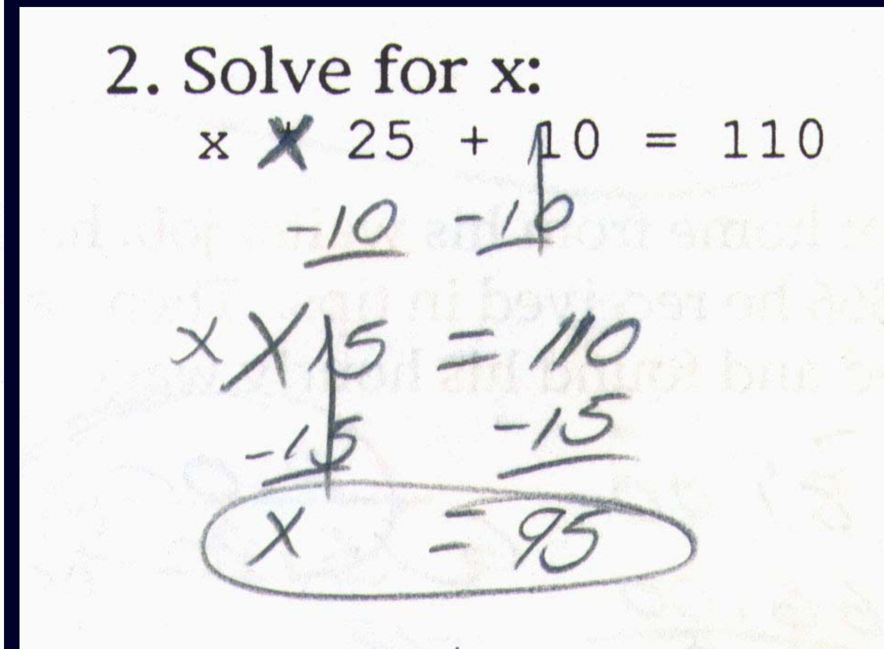


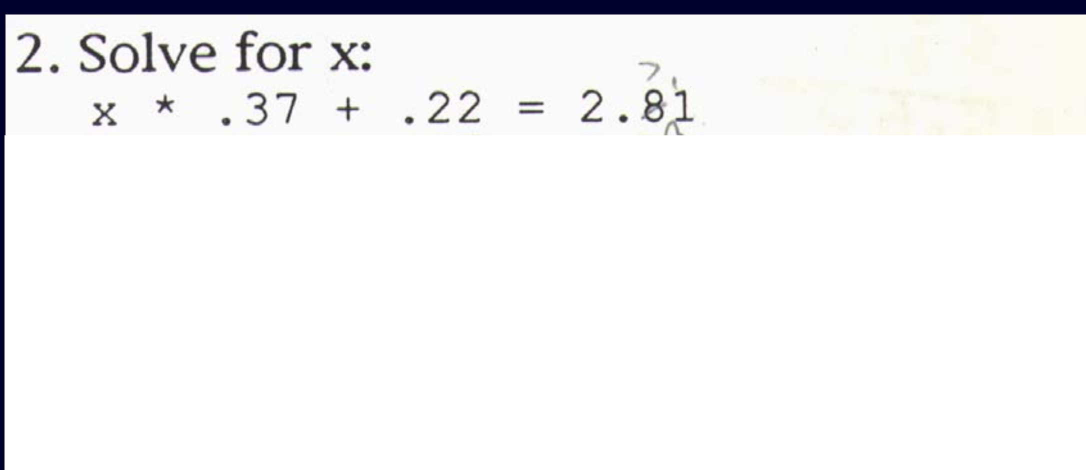
More no-response errors in equations

=> *Equation comprehension is hard*

Examples of Symbolic Comprehension Difficulties





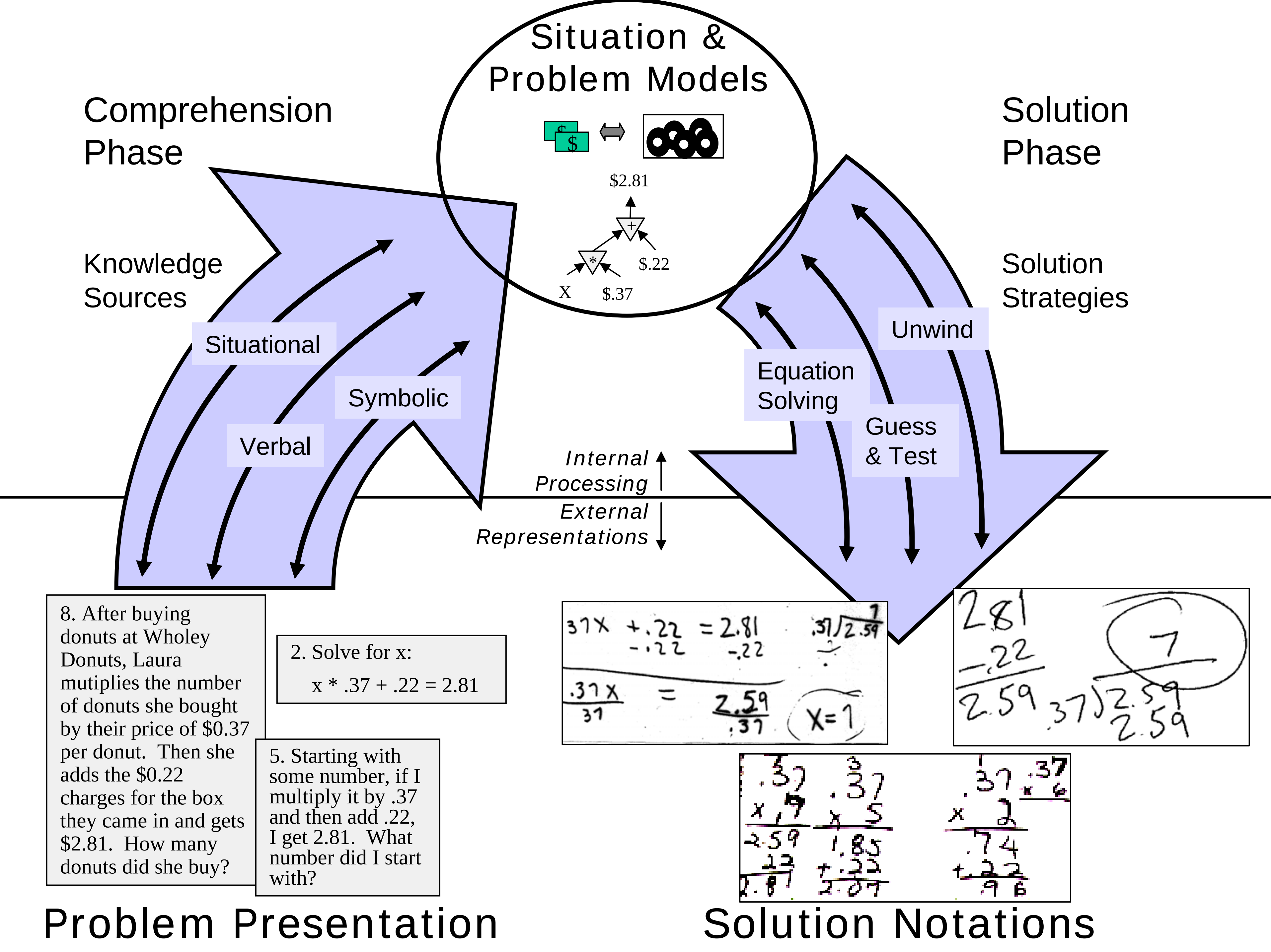


- * means times
- Both "sides"
- Order of ops
- No response

Why equations are harder despite prior beliefs & data

- In contrast to elementary, by junior high:
  - Students have better reading comprehension
  - Symbolic math demands increase
- We see verbal advantage for college students!
  - Particularly for less frequent symbolic forms like “(X - 64) ÷ 3 = 20.50” & “600 – 20 * x = 260”
- For higher complexity problems, story problems are indeed harder
  - Multiple unknowns: “X - 0.15X = 38.24” & “5.7X - 22 = 5.4X”

Koedinger, Alibali, Nathan. Trade-offs between grounded and abstract representations: Evidence from algebra problem solving.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Problem Presentation
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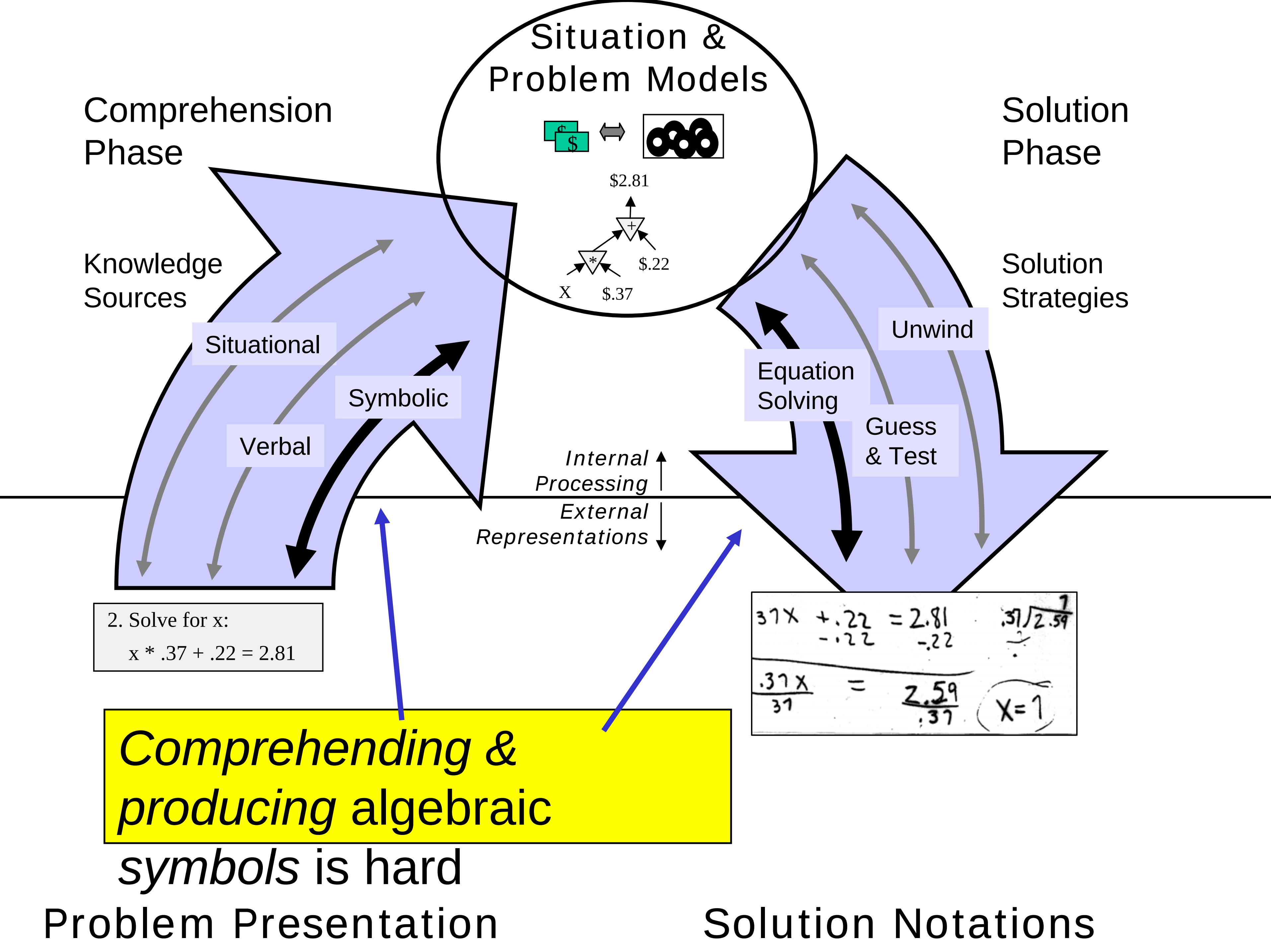

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Comprehending & producing algebraic symbols is hard
Problem Presentation
Solution Notations

Overview

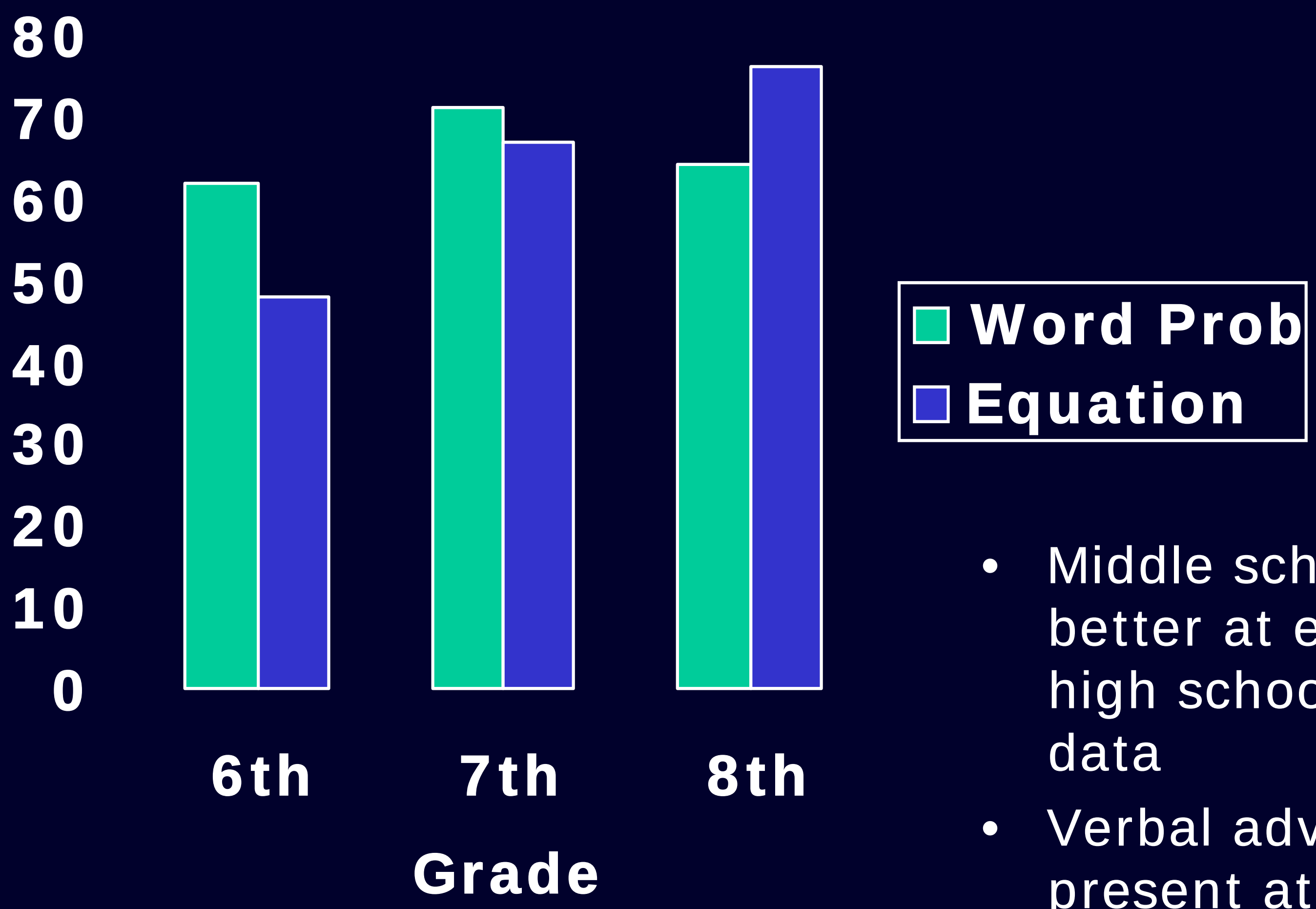
- Comparing matched verbal story & symbolic equations
- Analysis of strategies & errors
- *Teach algebra in middle school?*

- Bridging instruction studies

Some contrasting data from middle school students

- Prior data from 93-94, Pgh high school
- Pittsburgh adopted a new middle school curriculum
  - Introduces algebra in grade 6
- We performed matched comparison at middle school level
  - Used word-equations & equations

Middle School Algebra Yields Earlier Symbolic Competence



- Middle schoolers doing better at equations than high schoolers in prior data
- Verbal advantage still present at 6th grade

Overview

- Comparing matched verbal story & symbolic equations
- Analysis of strategies & errors
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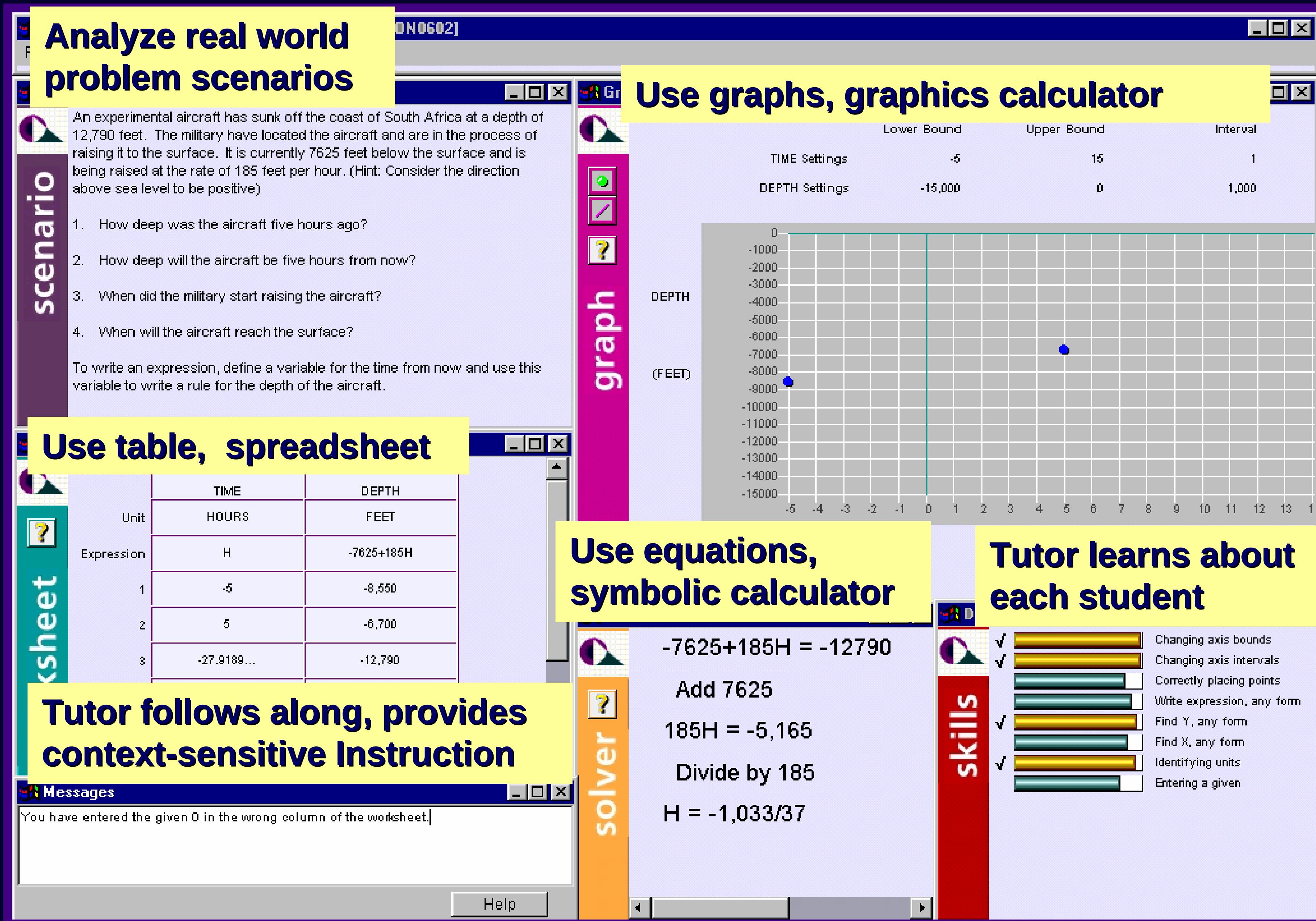
Cognitive Tutor Algebra

- Provides benefits of one-to-one tutoring
  - Computer-based tutor
  - Frees teacher to provide more individual help
- Based on cognitive science
  - Evaluations show enhanced student learning
- Full course used in some 2000 US schools with diverse populations

Algebra Cognitive Tutor Sample



Forester Textbook Problem

Drane & Route Plumbing Co. charges $42 per hour plus $35 for the service call.

1. Create a variable for the number of hours the company works. Then, write an expression for the number of dollars you must pay them.

→ Symbolize

2. How much you would pay for a 3 hour service call?

3. What will the bill be for 4.5 hours?

→ Arithmetic (find y)

4. Find the number of hours worked when you know the bill came out to $140.

→ "Algebra" (find x)

Inductive Support Version

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Arithmetic (find y)

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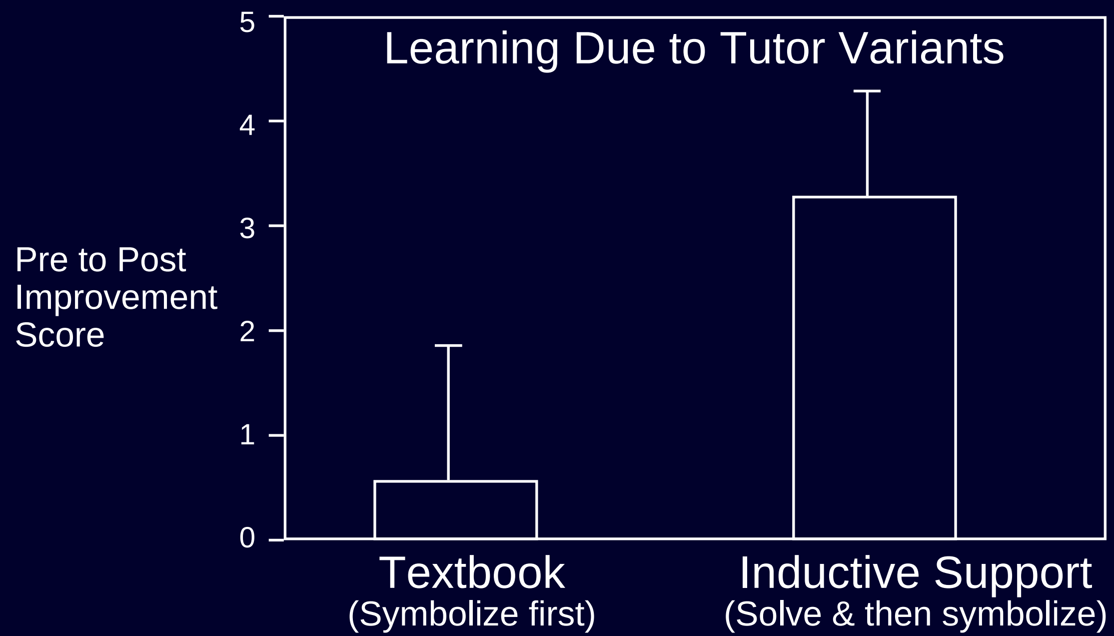
Symbolize

4. Find the number of hours worked when you know the bill came out to $140.

"Algebra" (find x)

Laboratory Study
Inductive Support Improves Learning



Textbook:

1. 35 + 42h = d

2. 35 + 42*3 = d
3. 35 + 42*4.5 = d

4. 35 + 42h = 140

Inductive Support:

1. 35 + 42*3 = d
2. 35 + 42*4.5 = d

3. 35 + 42h = d

4. 35 + 42h = 140

Koedinger, K. R., & Anderson, J. R. (1998). Illustrating principled design: The early evolution of a cognitive tutor for algebra symbolization. In *Interactive Learning Environments*, 5, 161-180.

Algebraic Function Instructional Study

Using a "bridging context" to build on prior informal knowledge

Teacher introduction:

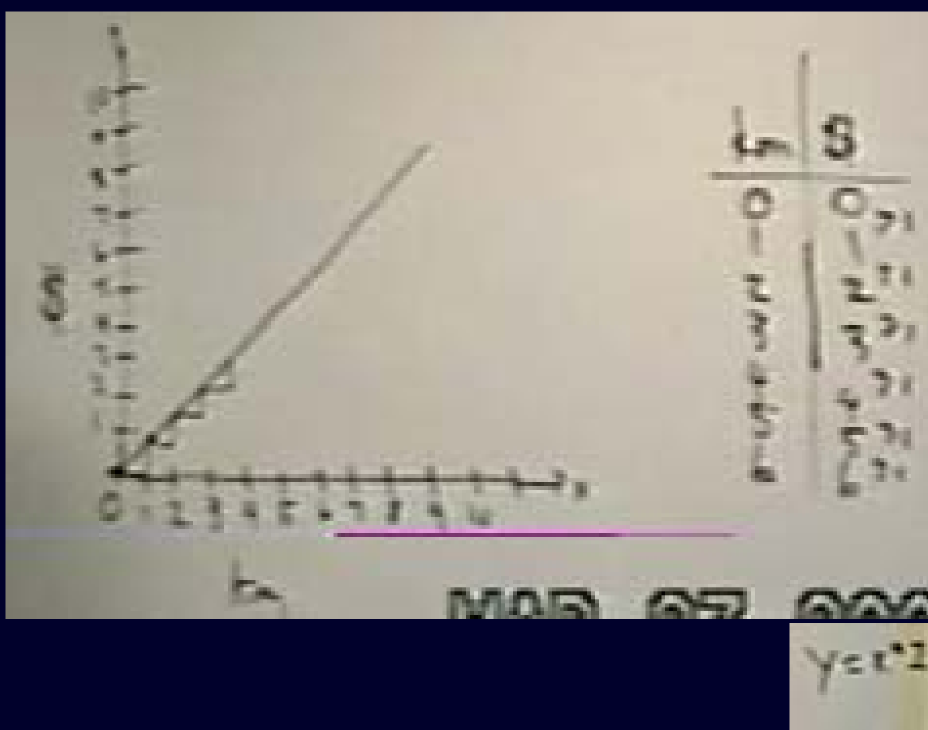
"We're looking at what we do to one set of numbers, to get other numbers. So how many of you have done something like a walkathon?
Say you're gonna sponsor me one dollar for every kilometer that I walk ..."

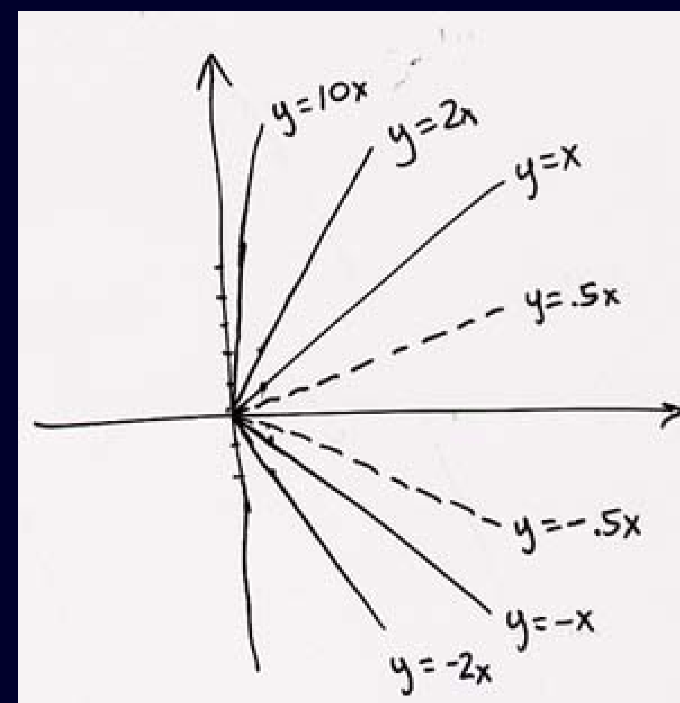
Kalchman, M. & Koedinger, K. R. (2005). Teaching and learning functions. In Donovan, S. & Bransford, J. (Eds.) *How Students Learn.* National Academy Press.

Multiple Representations & Deep Structure Contrast

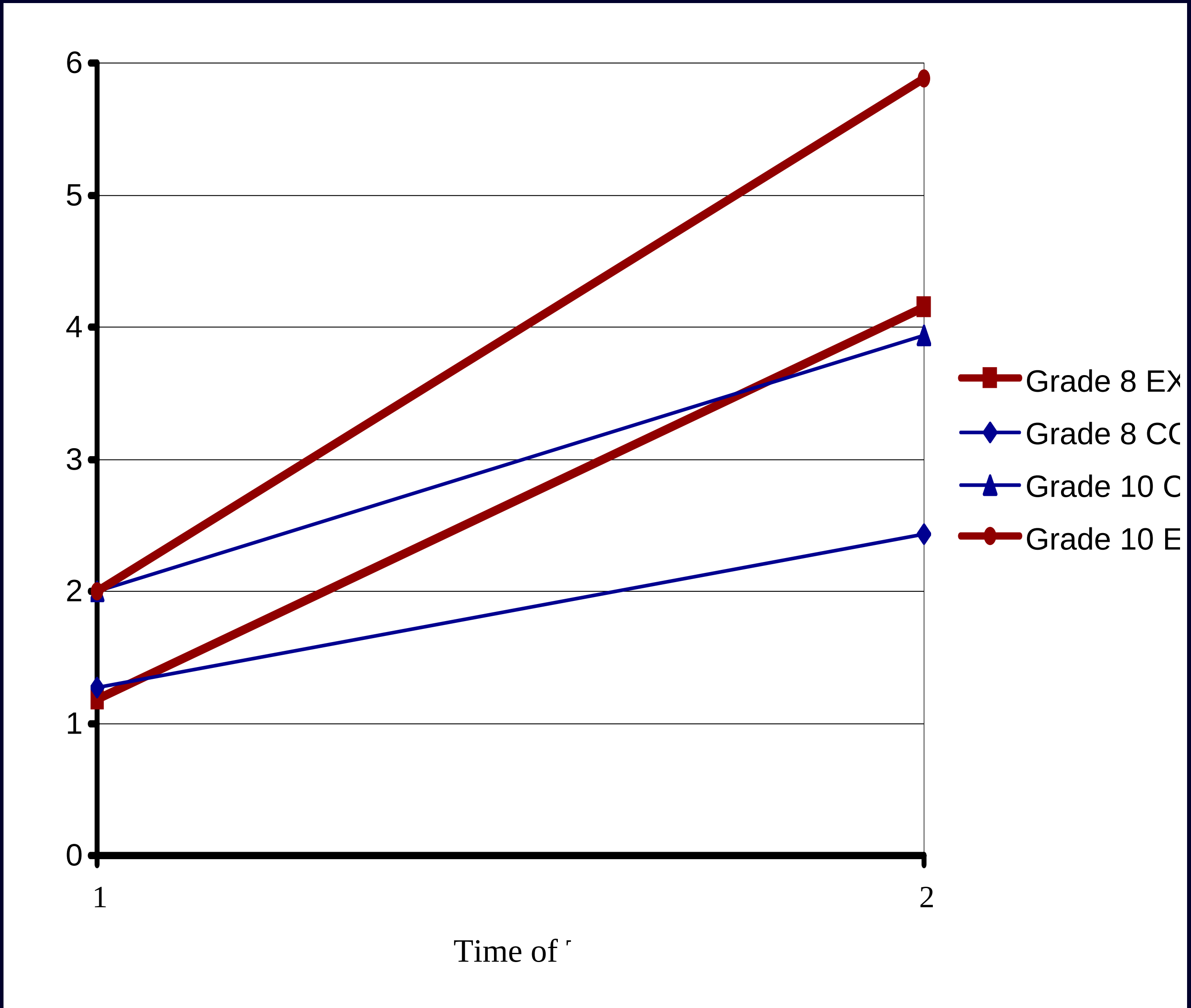
- Situation, words, table, graph, equation

- Maintain surface, situation
- Vary & compare deep function (deep structure)





Evaluation in Grades 8 & 10



Summary

- Students can reason about unknowns prior to formal algebra instruction
- Language of symbolic algebra is hard
- Early exposure to use of language reduces informal-formal gap, may improve instruction overall
- Bridging instruction, from informal to formal, can be effective

Related Research Efforts & Web Sites

- ACT-R cognitive modeling
  - Eye tracking, brain imaging using fMRI

actr.psy.cmu.edu

- Tools for authoring Cognitive Tutors
  - Variety of new domains: sciences, languages

ctat.pact.cs.cmu.edu

- “Assistments” for on-line dynamic assessment

assistment.org

- Pittsburgh Science of Learning Center

learnlab.org

THANK YOU!

Aims and Objectives of Algebraic Reasoning Conference

Although a burgeoning array of curricula for algebra and pre-algebra are being developed and marketed, comparatively few of these instructional approaches and materials are sufficiently grounded in rigorous research. Moreover, with some notable exceptions, pedagogical debates concerning the ability of primary, if not pre-K, children to acquire putative algebraic-relevant concepts (e.g., recognizing patterns, representing relationships, and making generalizations) are generally uninformed by the most recent theoretical and empirical advances in our understanding of cognitive development in general or mathematical cognition in particular. Concomitantly, the vast majority of research to date in the domain of mathematical cognition has tended to be rather narrowly focused on basic numerical and arithmetic processing, with comparatively little effort to extend this type of work to the study of the development of algebraic reasoning skills. This paucity of knowledge is particularly vexing in light of the documented complexities associated with the transition from arithmetic to algebra, as well as the conflicting perspectives proffered by various factions regarding the appropriate sequencing of mathematical content in the elementary and middle school years. For example, whereas some have argued that a solid grounding in traditional arithmetic principles and skills is crucial for learning algebra, others contend that the "algebrafying" of arithmetic in the early elementary grades will subsequently ease the transition to pre-algebra in middle school and basic algebra in high school.

In an effort to shed some light on this often heated controversy, the co-sponsors thought it would be timely to organize a conference focused on the developmental, cognitive, and disciplinary (i.e., mathematical) foundations for instruction in algebra, with the following general aims: a) examine what we know and don t know about the requisite developmental and cognitive competencies for proficient pre-algebraic and algebraic reasoning, and how best to address the gaps in our knowledge base; and b) analyze what kinds of math problems should (or should not) be categorized as algebraic in content from the perspective of the field of mathematics. Conference participants will possess a wide range of expertise, drawing from such diverse fields as developmental psychology, educational psychology, cognitive neuroscience, math education research, and mathematics. Through formal presentations, comments and questions periods, and plenary discussions, we are hopeful that extant theoretical perspectives will be evaluated in light of relevant empirical data, and in addition, that suggestions for future research priorities will emerge. In our view, a firm knowledge base will be crucial for informing the development of effective instructional approaches to improve the acquisition of algebraic concepts and skills.

Aims and Objectives of Algebraic Reasoning Conference

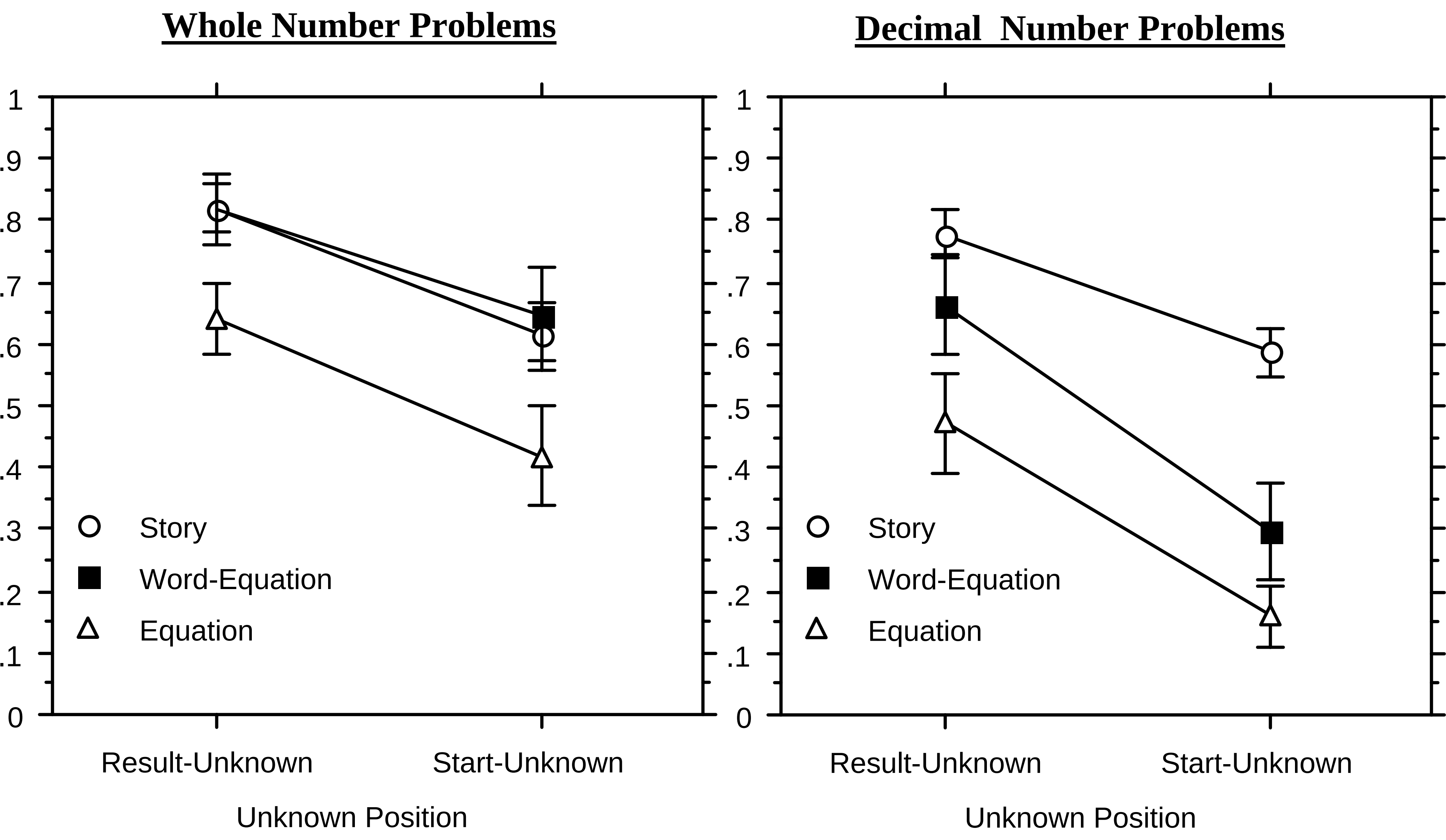
- Lots of curricula for algebra & pre-algebra, but not sufficiently grounded in rigorous research
  - Can primary children acquire algebra concepts?
    - recognizing patterns, representing relationships, making generalizations
    - uninformed by cog development or math cognition
  - Focus on basic numeracy misses
    - complexities in transition from arithmetic to algebra
    - grounding in arithmetic skills crucial for learning algebra vs. "algebrafying" of arithmetic in elementary grades to ease transition
- Conference: developmental, cognitive, & math foundations for instruction in algebra
  - What are requisite dev & cog competencies for proficient pre-alg & alg reasoning? Gaps?
  - What math problems should be categorized as algebraic from perspective of field of mathematics?

What Makes Algebra Hard for Learners?

To the surprise of most teachers and mathematics educators, we discovered that beginning algebra students are often better able to solve simple algebra word problems than the matched equations. In other words, often students can better reason algebraically when given a verbal description than a symbolic description of a mathematical problem. We have replicated this "verbal advantage" in multiple studies with different student populations in a variety of contexts. Analysis of student strategies and errors suggests, first, that students have informal algebra problem solving knowledge prior to acquisition of symbolic equation solving skills and, second, that key difficulties in learning algebra symbolism are essentially "language acquisition" issues. In controlled experiments with our technology-enhanced Cognitive Tutor Algebra course, we have demonstrated better student learning from instruction that bridges from of this prior informal knowledge to help students acquire the formal language of algebra. We have also found evidence in our replication studies for the idea that earlier introduction of algebraic symbolism in middle school helps ease algebra language acquisition difficulties.

EXTRA SLIDES ...

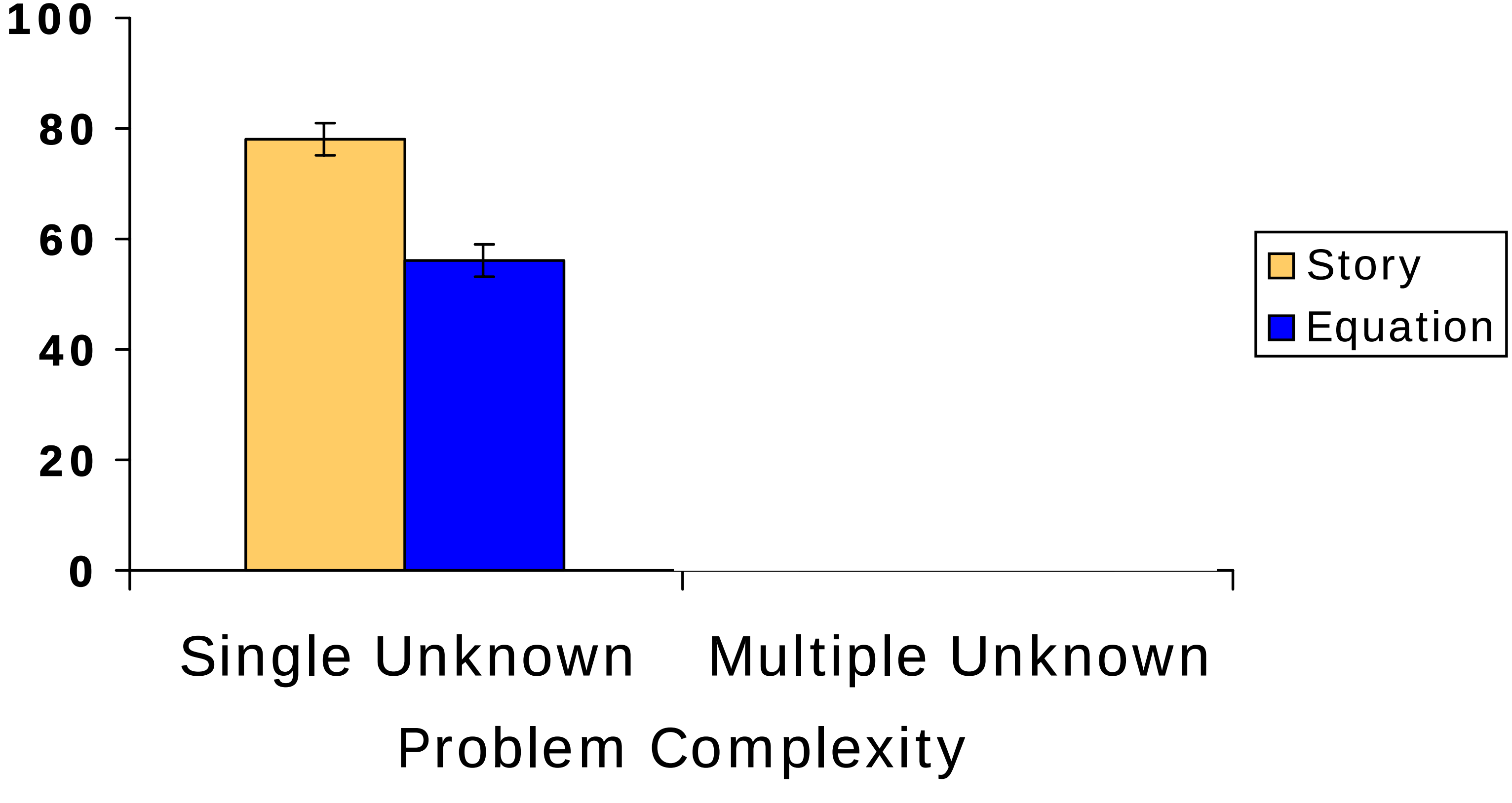
Main effects of all factors
Rep x number interaction



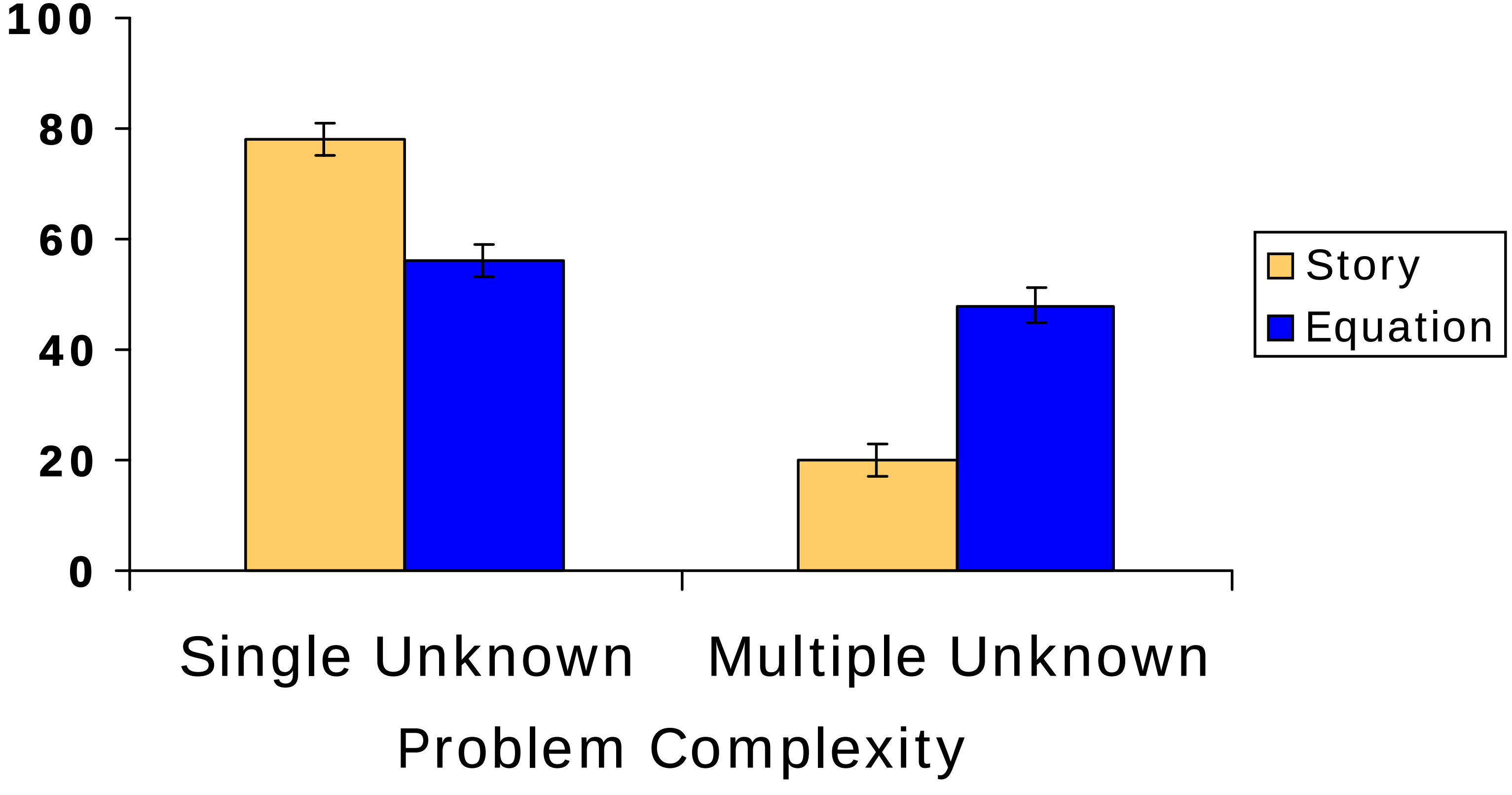
Which is harder for students: Story Problems or analogous Equations?

Single-unknown problems "Early" Algebra	**Multiple-unknown problems "Real" Algebra**
<u>Story Problem</u> Mom won some money in a lottery. She kept $64 for herself and gave each of her 3 sons an equal portion of the rest of it. If each son got $20.50, how much did Mom win? <u>Equation</u> $(X - 64) \div 3 = 20.50$	<u>Story Problem</u> Roseanne just paid $38.24 for new jeans. She got them at a 15% discount. What was the original price? <u>Equation</u> $X - 0.15X = 38.24$

College Students in a Remedial Algebra Class



On Complex Problems: Abstract Reps are Easier



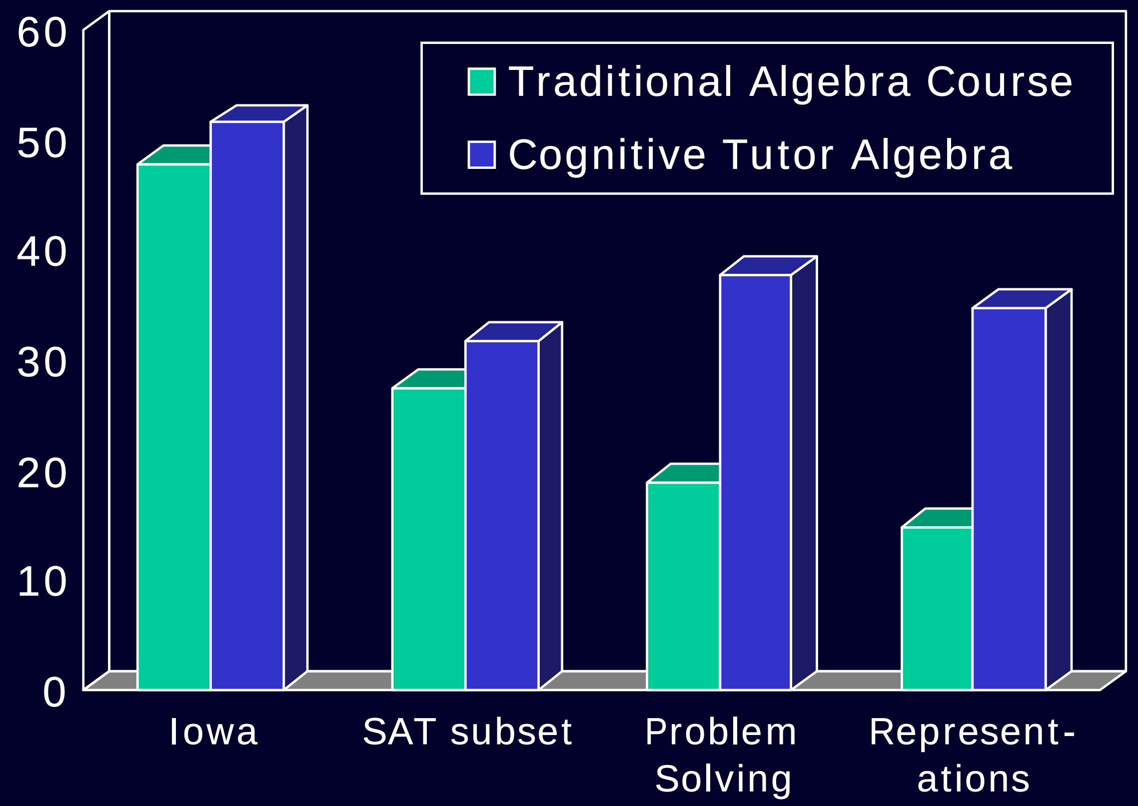
Research Shows Cognitive Tutors Work!

- Full year classroom experiments in city schools
- Many studies (see carnegielearning.com)

- Typical Results:

  50-100% better on problem solving & representations

  15-25% better on standardized tests



Koedinger, Anderson, Hadley, & Mark (1997). Intelligent tutoring goes to school in the big city.

Research-based Instructional Design Commitments

- need to:
- - decompose knowledge not materials or behaviors!
- - need to be aware that knowledge includes retrieval strategies and patterns (if-parts) as well as facts, reasoning processes, and operations (then-parts)
- - consider development issues, intermediate partial knowledge states