Comments on NAEP Algebra Problems

Roger Howe 14 September, 2005

I. General Discussion

What is Algebra?

The question posed is, are the NAEP questions which are classified as being about algebra, indeed about algebra, or about pre-algebra, or algebra readiness, or arithmetic, or yet something else? To discuss this, one should start with at least a preliminary description of algebra. I understand algebra to be primarily about two main topics.

i) Working with variables, and in particular, arithmetic with variables, so the formation of polynomial and rational expressions.

This also includes representing, or "modeling" concrete situations with expressions, and setting up equations. It is also often extended to include extracting roots. (If these processes are iterated, they can produce highly complicated expressions. But school algebra does not go very far down this road.) It also includes manipulating expressions and equations, to simplify, solve and interpret.

ii) Algebraic structure, primarily as captured in the Rules of Arithmetic (aka, the field axioms).

The Rules of Arithmetic encapsulate the legal manipulations on polynomial or rational expressions. If taking rational powers is allowed, they have to be supplemented by the Law of Exponents and the multiplicativity of fractional powers. These rules, together with the principles for transforming equations (the original techniques which gave rise to the subject known to us as algebra), summarize the basis for algebraic technique, which works on the algebraic expressions described in i), and on equations between them.

In recent years, "algebra" has been construed somewhat differently by some mathematics educators, and this is reflected in NAEP. In particular, the study of "patterns" has been declared by some to be algebra. I am skeptical that this has been productive. What a "pattern" might be, and what might be a pattern, is not specified, but in practice a rather restricted collection of functional relations, predominantly first order, are what get called "patterns". These patterns have some claim to be connected with algebra, but identifying these and a few others as "patterns" trivializes the term. At the same time, many highly significant patterns, of which the most important are captured in the Rules of Arithmetic, get inadequate attention.

Coordinate geometry is distinct from algebra, but is often tightly linked with it, especially pedagogically, since it provides a visual approach to equations, especially linear equations. There are a number of NAEP problems which concern coordinate geometry. At the 4th grade, these all deal with the basic yoga of coordinatization. However, at the 8th grade, there are 3 problems which are at a somewhat higher level, including one problem which asks for a solution of a graphically given system of two linear equations.

The NAEP Problems, Classification

Using the above description of algebra, I would classify the NAEP problems I was presented with as follows.

	Algebra	Prealgebra	Arithmetic	Patterns	Implicit Patterns	Coordinates
4th Grade:	5	4	1	4	5	3
8th Grade:	18	2	4	3	6	3

In addition, at the 8th grade, there are three problems which probe understanding of coordinate geometry beyond basic familiarity with coordinatization. There are also two problems which test basic logical reasoning.

This classification says that algebra is a minor part of the 4th grade question set, but that at 8th grade, a plurality of problems involve algebra.

Some remarks on this classification:

a) Since algebra is inherent in arithmetic, and even our decimal (base ten, place value) system for writing numbers exploits algebraic structure heavily, the line between algebra and arithmetic is fuzzy. Accordingly, the above classification does not lay any claim to being definitive, and probably no definitive classification is possible.

b) The category of "Implicit Patterns" needs explication. Adherents to the doctrine that patterns are the key to algebra often want students to perceive a pattern which is unspecified but can be extracted from the problem. Problems of this sort are here labeled as Implicit Pattern problems. They are common on NAEP, being the largest category at 4th grade. The total of patten problems, combining the Implicit Pattern problems with well-defined problems involving patterns, accounts for about 40% of the 4th grade set, and over 20% of the 8th grade set. Problems of this sort can serve a useful pedagogical purpose in the classroom, but there is a widespread and strong aversion among mathematicians to having such problems on tests. The feeling is that these problems amount to asking the student to guess what is in the mind of the poser. If one were to lump all the pattern problems with the Algebra, then Algebra would be the dominant topic at both grade levels.

c) If one likewise lumps the coordinatization problems in algebra, and also the coordinate geometry problems at 8th grade, then one gets a classification as follows:

4th Grade:	Algebra + P + IP + C	Prealgebra	Arithmetic
	17	4	1

8th Grade:	Algebra + P + IP + C + CG	Prealgebra	Arithmetic	Logic
	33	2	4	2

d) The problems which test logical reasoning are among the nicest problems on the 8th grade exam. On the other hand, they are relatively simple problems of their type.

e) More important to me than the bins in which the problems fit is the overall level of the problems. I would say that most of these problems are very basic and not interesting. By comparison, the Singapore 5th and 6th grade textbooks contain problems which, although they are considered part of arithmetic, are more difficult than any of these NAEP problems. Indeed, some of the more challenging and interesting problems, which I classified as algebra, could be, and probably would be in Singapore, listed as arithmetic problems. Examples include problem # 2 on both 4th and 8th grade NAEP. I labeled this as algebra, but it can be solved by elementary reasoning requiring no algebraic technique. Likewise, problem # 10 on the 8th grade exam (which was among the most difficult problems), was labeled algebra, because it can be formulated as system of two linear equations. However, it is merely an arithmetically complicated version of the chickens and horses, feet and heads problem, and can be solved by simple reasoning without writing any equations, or even by guess and check. These remarks echo those of a recent AIR report comparing Singapore and US mathematics education (What the United States Can Learn From Singapore's World-Class Mathematics System, prepared by the American Institutes of Research for the U.S. Department of Education, January 28, 2005).

Things which make the problems easy include a predominance of one step and two step problems. Excluding the Implicit Pattern problems, which are hard to classify in terms of number of steps, there are no problems requiring more than 3 steps, and very few of these. Also, in most of the problems classified as algebra, the arithmetic is extremely simple.

There are no problems which involve substantial or even moderate manipulation of equations. For example, there are no problems which ask to simplify an expression containing variables using order of operations. (At grade 4, there is one problem testing facility with order of operations (#4), and two at grade 8 (#12, #24). They are arithmetic expressions (involving only numbers); the expressions are fairly simple, and the numbers are also easy numbers.) There are no problems which require expanding a product of binomials. There are no problems which involve simplifying an expression by distributing a negative sign. The systems of linear equations which appear are of very restricted type, and can be dealt with using <u>ad hoc</u> techniques.

There are no problems which require dealing with a quadratic expression. In fact, quadratic expressions are virtually absent. There are two Implicit Pattern problems (#24, # 31 at 8th grade) which involve modeling by quadratic functions. Both are poorly formulated, even for Implicit Pattern problems. There is one fourth grade problem (# 22) which asks for evaluation of a very simple quadratic function described verbally. There are no problems which involve solving a quadratic equation.

Since all these fairly basic technical issues are not probed, it is clear that there are no problems which require combining two or three such skills.

Besides these technical issues, understanding of variables and how to use them is not probed deeply. Only very simple situations are required to be turned into expressions.

f) The common problem # 13 (4th grade), or # 21 (8th grade) involves a described pattern. I have classified it as a Pattern problem rather than an Implicit Pattern problem because I regard the verbal description plus the accompanying illustration as sufficient to specify a particular function. However, this problem was among the hardest, as measured by % correct, at both the 4th grade and 8th grade level. Since the algebraic expression involved ($x+1$) is one of the simplest imaginable, the observed difficulty level must stem from poor problem formulation.

Given the level of these problems, I do not see how NAEP could claim to determine that any student is "proficient" in algebra. How much algebra should be examined on the 8th grade NAEP, given the current state of curriculum in the US, is a question for debate. However, one can safely say that these questions do not provide a thorough probe of Algebra I.

II. Discussion of Individual Problems

4th Grade:

There are 22 problems, which fall into several groups.

There are several problems of the "implicit pattern" type: some numbers or configurations which are considered to exhibit a "pattern", not otherwise specified, are displayed. The student is supposed to discern the "pattern" and extrapolate it in some fashion, according to the instructions of the particular problem. Such problems include # 6 (a "pattern" (of constant differences) in a table of cricket chirp rates at various temperatures), # 10 (a repeating pattern or design), # 12 (a "pattern" of weight gain of a puppy), # 14 (a "pattern" of powers of 2), and # 17 (a "rule" used to define a table of values). It is a widely held opinion among mathematicians that these problems are not mathematics (in particular, not algebra) and should not be used on tests, especially standardized tests (although they may have a valid use in classroom discussions). Questions in which a "pattern" is not specified, but is left to the student to discern are basically exercises in mind-reading, not reasoning. The fact that many students learn how to answer such questions successfully bears witness to the impoverished collection of pattern types which are treated in such problems.

Problem # 12 is especially troublesome. In it, a puppy's weight each month is given. The difference between successive months decreases by one each month. This is the "pattern" which the student is supposed to observe. But this "pattern" in a table would normally be considered to signal a quadratic function, which after reaching its maximum (in months 6 and 7) would then start decreasing, faster each month, eventually going to $-\infty$. Is this the "pattern" that is supposed to govern the puppy's weight? Or is a quadratic function supposed to govern through its maximum only, and then the weight becomes constant? What sort of a "pattern" is this supposed to be?

Problem # 10 is quite simple - a basic U or C shape, and its rotates by multiples of 90°, are arranged along a line. One could construe the shapes as repeating every four places, as is intended. The student is supposed to guess this intention, and then use that to deduce the shape belonging in the next place after the end of the ones shown. It was answered correctly by 91% of students. This does not mean that it is a properly formulated problem. There is no logical basis for choosing the intended answer over the others.

Problems # 14 and # 17 are similarly flawed: there is no basis for deducing the desired answer; one must guess. Besides philosophical objections to such problems, there is a practical objection: in order for such problems to be answerable by a reasonable number of students, they must trade in very simple, readily discernable "patterns", chosen from a very limited repertoire. Thus, they essentially become template problems: take your small list of government-issue patterns and see which one fits the situation. Such problems over time become more rote than basic arithmetic.

There are a number of other problems involving patterns which do not suffer from the defect that students must guess the pattern. These include problems # 1, # 13, # 15, # 16, # 20.

In problem # 1, the student is told that s/he is looking at a problem that "repeats in groups of 3", which is probably an age-appropriate way of saying that it is periodic with period 3. It is then clearly possible to determine the missing letters by looking at a place 3 steps away from the blank places. One must look backward for one, and forward for the other. I think this is a pleasing problem requiring some simple reasoning. I would not call it algebra.

In problem # 13, a pattern is established pictorially and by verbal description. There still is some ambiguity in the problem, but not as serious as in the five in the first group. This could be dealt with as an algebra problem, but that would be a pretty heavy treatment. I would call this one pre-algebra or arithmetic. It involves an insight: the first picture requires two tacks, and each new picture requires only one new tack, so there is one more tack than the number of pictures. This was a hard problem - only 25% correct.

Problem # 15 gives a rule for year-on-year increase in enrollment, and asks to compute the consequences of this rule for three successive years. Again, I would construe this as arithmetic or perhaps pre-algebra. Finding an expression for the year $1990+n$ in terms of n could be considered algebra.

Problem # 16 is awkwardly worded, but is saved from disaster by being multiple choice. Students only have to check the given rules against the numbers in the table to see which ones don't work. Choice C) can not be eliminated by checking the first pair. Choice B) is the inverse of A), which is the answer. Choice D) is the inverse of C). This is basic algebra, in the sense of working with expressions.

Problem 20 is another repeating pattern problem. There seems to be only one way to interpret the information give, so the problem is well-defined. Again, I would not call it algebra. It is pattern recognition, requiring some logical reasoning.

Other Problems.

Problem 2 presents a pictorial/physical equation, and asks for a solution. The problem translates easily into algebra, although it can be done by ad hoc reasoning. It has an algebraic feel to it. I would call it algebra or pre-algebra.

Problem 3 involves understanding inequality. The multiple choice format resolves possible uncertainties in what is meant by "whole number". This is arithmetic, except that it uses a variable in its statement, so it could be called algebra.

Problems 4 and 5 involve the same structure. They could easily be formulated in terms of variables (B = number of bicycles; W = number of wagons; w = number of wheels; $w = 2B + 4W$). However, it clearly can also be done by arithmetic. Problem 4 is s direct problem, and problem 5 is an inverse problem, involving some implicit equation solving. This is reflected in the percent correct: 69% and 29% respectively. Probably pre-algebra is a good classification, or tending towards arithmetic for # 4 and towards algebra for # 5.

Problem # 7 asks for substitution for variables. This is algebra of a basic sort.

Problems # 8, # 18 and # 19 involve the rudiments of coordinate geometry - basically, the conventions of coordinatization. This is not strictly algebra, but it is typically taught intertwined with algebra, since is provides a way to visualize linear functions.

Problems # 9 and # 11 involve identifying simple algebraic expressions. This is rudimentary algebra.

Problem #21 asks for a missing factor. It could be interpreted as solving the equation $17x = 204$, so it might be considered rudimentary algebra. But it is also accessible by arithmetic.

Problem # 22. Here a function is given in verbal form, and a student is asked to evaluate it at a given point. This can be considered as basic algebra, but variables are not really eseentially engaged, so it also has an arithmetic air.

General remark about the tables: The formatting of the tables in the version of the test supplied to me is awkward and hard to read. This applies to problems # 6, # 12, # 15, and # 22. I don't know if students were presented with this formatting. I hope not.

8th Grade:

Problems # 1, # 2, # 3, # 22, # 23, # 24, # 25 and # 30 are respectively the same as problems # 1, # 2, # 3, # 10, # 11, # 12, and # 13 and # 18 from the 4th grade. It is a relief to see that the percentage correct increases substantially from 4th to 8th on all problems. Interestingly, the most troublesome problem at 8th grade is problem # 25/13, which is not technically difficult. To my reading, problem # 24/12 seems both more obscurely formulated (it is an unspecified "pattern" problem), and more demanding technically. It was slightly more difficult than # 25/13 at fourth grade. This change in relative performance may indicate either a defect in the formulation of # 25/13, or that problems of the type of # 24/12 have become template problems.

New "implicit pattern" problems at 8th grade are # 7, # 28, # 31, # 35 and # 36. Formulation aside, problem # 7 is a nice problem, in that it requires equating a fraction with a decimal. In principle it is algebra: it can be thought of as asking for the solution of $\frac{n}{n+1} = .95$. However, the formulation in terms of "patterns" detracts from the algebraic nature of the problem, which can also be dealt with by guess-and-check on a calculator: just plug in $\frac{n}{n+1}$ for various n until you get .95.

Problem # 28 is about arrays of dots, a mainstay of "pattern" problems. It is poorly worded, in that "pattern" is used in two different senses: it refers to the individual arrays of dots, and to the rule by which they change from step to step. Also, the second sentence of the problem statement provides little or no information, it just adds to the reader's burden. (Or, one could interpret this as being the salient information in the problem, and solve accordingly. This would lead to a qualitatively different answer from the one intended.) This is quite a poor problem. It is not surprising that the percent correct (5% , plus 1% for partial credit)is so low.

Problem # 31 is an "extend the table" problem, using an unspecified "pattern", in this case the linear function $n \to 2n + 1$. If the function had been specified, then it could be classified as an algebra problem of a basic type - evaluate a function at a given point.

Problems # 35 and # 36 are also dot-array problems. At least in problem # 35, the individual arrays are called "dot-figures", and "pattern" is reserved for the rule by which the figures grow. From an algebraic point of view, it is identical to problem # 31, in that the size of the n-th dot pattern is $2n + 1$. I can not interpret problem # 36. There seems to be a misprint. In any case, it does not ask for an answer, but for an explanation of an answer. It not surprisingly has a very low success rate (14 %)¿

Besides the problem # 30, repeated from the 4th grade exam, problems # 5, # 12, # 14, # 15, and # 41 involve coordinate geometry. Aside from problem # 41, which is a basic plotting problem, analogous to the 4th grade problems, these questions are at a higher level than the 4th grade problems.

Problem # 5 in principle involves some reasoning using the parallelism of the sides of a square: one should check that the x-coordinates of the points A and D are the same, and reason that therefore, the x-coordinates of points B and C must be the same; and likewise for y-coordinates. However, probably students will just take their cue from the figure, and decide that the x-coordinate of C is the same as the x-coordinate of B, because it appears to be; and likewise for the y-coordinate. Problem # 5 would be much more interesting if the sides of the square were not parallel to the axes.

Problem # 12 is an interesting problem, in that it asks for a qualitative statement about a line specified by two points. However, it suffers from unfortunate wording. It asks which of 5 choices "must be" true (emphasis in original). In mathematics, there is no difference between "must be" and "is". Sometimes we say "must be" when we make need only a small amount of the available data to make a prediction, and do not explicitly verify it. But "must be" is not good test-problem language, particularly the must.

Problem # 14 is a reasonable problem, asking students to deduce equations of two lines from their graphs, and then find the point of intersection. Its main defect from the point of view of algebra is that it is subject to graphical and guess-and-check solutions, as well as the straightforward algebraic one. It also suffers from poor wording, saying that the rates represented by the slopes of the lines will "continue as shown" rather than the more precise and economical "remain constant". The problem asks for an explanation of method as well as the answer. One unfortunately cannot tell which part is responsible for the low (19 %) success rate.

Problem # 15 is a reasonable problem, implicitly involving vector addition. Again, however, the problem formulation, requiring location of the treasure chest as an intermediate step, reduces the formal technical burden of the problem, since after the chest location is marked, one can just count to find its location relative to the tree.

Problems # 13 and # 29 test understanding of order of operations conventions. Although these are important for algebra, these questions are about arithmetic. There seems to be a misprint in # 29.

Problem # 21 requires integer arithmetic (subtraction of signed numbers). It is a little artificial, but the low success rate (22 %) is surprising, particularly in view of the concrete context.

Problems # 6, # 8, # 9, # 11, # 16. # 18, # 19, # 33, # 34, # 37, # 38, # 39 can be considered to be about basic algebra. Problem # 6 asks to check algebraic expressions against a table. Problem # 8 tests knowledge of the definition of average. It is a bit forced. Problem # 9 asks for a very simple manipulation. Problem # 10 is a problem which can be considered algebra, but is also solvable by arithmetic with added reasoning, and by guess-and-check. It has some of the more demanding arithmetic on the test. Problem # 11 involves determining a variable x, which is constrained partly by stated conditions, and partly by the choices offered. Problem # 16 uses a variable, and requires understanding and translating into symbolic form the fact that consecutive even numbers differ by 2. Problem # 18 asks to check whether given coordinate pairs satisfy a linear equation in two variables. Problem # 19 asks for symbolic translation of a verbal statement. Problem # 44 asks for the smallest whole number solution of a simple inequality, in a multiple choice format. A constructed response format would have been more probing here. Problem # 34 is a contextualized version of problem # 9. I think it is a good idea to ask such parallel pairs of questions. Problem # 37 is formally a very simple equation. It may be considered algebra or arithmetic. It can be readily approached by guess-and-check. It had a very high (89 %) success rate. Problem # 38 is a somewhat harder linear equation in one variable. It could also be done by guess and check, but backtracking might be more effective. Problem # 39 gives three ordered pairs, and asks for a rule giving the second entry in terms of the first. It is in multiple choice format. It is a better formulated variant of problem # 31. It had a moderate (48 %) rather than poor (25 % for # 31) success rate, but still proved quite challenging.

Problem # 27 asks students to recognize the infinitude of numbers. Technically, it is algebra, because it asks about the values of an expression.

Problem # 20 involves what is sometimes called “algebraic thinking”. It states a fact obout diagonals of polygons for 4-, 5-, 6- and 7-sided polygons, and asks for the corresponding fact about 20-sided polygons. It could be handled by algebra, but seems more likely to be done by pattern recognition. I wish that the polygons had been specified to be convex; without this, the notion of “diagonal” becomes problematic.

Problems # 17 and # 32 are basic logic problems. They are nice enough problems, but not algebra.

Problem # 4 iasks to locate on the number line a set defined by upper and lover bounds. I don’t know how to classify this.

III. Problem Difficulty and Summary Description

4th Grade

Number	% Correct	Type	Description
1	52	Pattern	Fill in repeating pattern
2	39	Arith/Alg	Pictorial Algebra
3	24	Arithmetic	Solve Inequality
4	69	Arithmetic	Compound Computation
5	29	Prealgebra	Solve Indeterminate Equation
6	09	Implicit Pattern	Extend Implicit Linear Relation
7	77	Algebra	Evaluate by Substitution
8	44	Coordinatization	Plot Points
9	67	Algebra	Select Algebraic Expression
10	91	Implicit Pattern	Extend Implicit Periodic Pattern
11	48	Algebra	Select Algebraic Expression
12	32	Implicit Pattern	Extend Implicit Numerical Pattern
13	25	Pattern	Extend Described Pattern
14	27	Implicit Pattern	Extend Implicit Numerical Pattern
15	91	Arith/Prealg	Perform Recursive Calculation
16	42	Algebra	Select Rule Fitting Table
17	24	Implicit Pattern	Extend Implicit Numerical Pattern
18	38	Coordinatization	Plot Points Satisfying Condition
19	90	Coordinatization	Determine Coordinates of Point
20	74	Pattern	Extend Repeating Pattern
21	69	Arith/Prealg	Find Missing Factor
22	15	Prealgebra	Execute Given Procedure

8th Grade

Number	% Correct	Type	Description
1	65	Pattern	Fill in repeating pattern
2	75	Arith/Alg	Pictorial Algebra
3	63	Arithmetic	Solve Inequality
4	45	Number Line	Locate Set with Geven Bounds
5	62	Coordinate Geometry	Locate Point
6	45	Algebra	Identify Linear Equation from Points
7	27	Implicit Pattern	Locate Number in Sequence
8	58	Algebra	Know Definition of Average
9	48	Algebra	Select Algebraic Expression
10	15	Algebra	Solve Linear Equations
11	77	Algebra	Select Value Solving Inequality
12	44	Coordinate Geometry	Qualitative Property of Lines
13	52	Arithmetic	Use Order of Operations
14	19	Algebra	Solve Equations Given Graphically
15	40	Coordinate Geometry	Locate Point via Vector Addition
16	45	Algebra	Identify Algebraic Expression
17	43	Logical Reasoning	Solve Logic Puzzle
18	41	Algebra	Identify Solution to Linear Equation
19	58	Algebra	Identify Expression from Description
20	54	Pattern	Extend Described Pattern
21	22	Arithmetic	Subtract Signed Numbers
22	96	Implicit Pattern	Extend Implicit Repeating Pattern
23	81	Algebra	Identify Expression from Description
24	66	Implicit Pattern	Extend Implicit Numerical Pattern
25	48	Pattern	Extend Described Patttern
26	40	Arithmetic	Provide Solutions to Inequality
27	72	Algebra	Know Infinitude of Expression Values
28	06	Implicit Pattern	Execute Implicit Array Pattern
29	22	Arithmetic	Use Order of Operations
30	75	Coordinatization	Plot Points Satisfying Condition
31	25	Implicit Pattern	Extend Implicit Numerical Pattern
32	62	Logic	Solve Logic Puzzle
33	44	Algebra	Identify Value of Variable Solving Inequality
34	14	Prealgebra	Identify Expression from Description

35	34	Implicit Pattern	Extend Implicit Array Pattern
36	14	Algebra	Explain Solution
37	89	Algebra	Solve Simple Equation
38	69	Algebra	Solve Linear Equation
39	48	Prealgebra	Identify Procedure from Ordered Pairs
40	45	Algebra	Use Formula to Complete Table
41	30	Coordinatization	Plot Point from Ordered Pair