Sticky Prices and Variable Markups

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1 Introduction

This note describes the solution of a simple and standard New Keynesian model with Calvo sticky price adjustment. The approach is to find the exact solution to a perfect-foresight version of the model. I avoid the reduced-form log-linearization generally used for this class of models, not because the solution is more accurate, but because the underlying principles of the model are much clearer in the exact version.

Experience has shown that the solution to the perfect-foresight model, starting just after a shock has displaced the model from its stationary point, provides a good approximation to the expected path of a full stochastic model perturbed by the same shock. A full solution to the stochastic model requires much more advanced numerical methods. With a small shock, the method used here gives essentially the same results as the standard log-linearization.

2 Model

Competitive firms produce output y from intermediate products $\tilde{y}_\tau$ using the CES technology,

$$y = \left(\int_0^1 \tilde{y}_\tau^{\frac{\epsilon}{\epsilon-1}} \right)^{\frac{\epsilon-1}{\epsilon}}, \tag{1}$$

with $\epsilon > 1$. The prices of intermediate products are $\tilde{p}_\tau$. The price of the final output is the CES unit cost:

$$p = \left(\int_0^1 \tilde{p}_\tau^{-(\epsilon-1)} \right)^{-\frac{1}{\epsilon-1}}. \tag{2}$$

A single monopolist produces each intermediate product. Calvo's randomly sticky price model constrains the price of an intermediate product. When a producer resets its price, $\tilde{p}$, that price remains in effect until the next time it is allowed to reset, an event that occurs with probability $1-\theta$. The demand for the firm's product is

$$y\left(\frac{\tilde{p}}{p}\right)^{-\epsilon}. \tag{3}$$

The firm chooses $\tilde{p}$ to maximize expected discounted future profit from sales governed by that price:

$$\sum_{\tau} \theta^{\tau} m_{t,t+\tau}\left(\frac{\tilde{p}_t}{p_{t+\tau}} - z_{t+\tau}\right) y_{t+\tau}\left(\frac{\tilde{p}}{p_{t+\tau}}\right)^{-\epsilon}. \tag{4}$$

Here $m_{t,t+\tau}$ is the economy's real discounter (the marginal rate of substitution between consumption in period t and consumption in period $t+\tau$) and z is the marginal and average real cost of producing a unit of an intermediate product. I write the present value as

$$A_t \tilde{p}_t^{-(\epsilon-1)} - B_t \tilde{p}_t^{-\epsilon}, \tag{5}$$

where

$$A_t = \sum_{\tau} \theta^{\tau} m_{t,t+\tau} y_{t+\tau} p_{t+\tau}^{\epsilon-1} \tag{6}$$

and

$$B_t = \sum_{\tau} \theta^{\tau} m_{t,t+\tau} z_{t+\tau} y_{t+\tau} p_{t+\tau}{}^{\epsilon}. \tag{7}$$

These two quantities obey the backward recursions,

$$A_t = y_t p_t^{\epsilon-1} + \theta m_{t,t+1} A_{t+1} \tag{8}$$

and

$$B_t = z_t y_t p_t^{\epsilon} + \theta m_{t,t+1} B_{t+1}. \tag{9}$$

The first-order condition for maximizing discounted future profit is

$$-(\epsilon-1) A_t \tilde{p}_t^{-\epsilon} + \epsilon B_t \tilde{p}_t^{-\epsilon-1} = 0 \tag{10}$$

or

$$\tilde{p}_t = \frac{\epsilon}{\epsilon-1} \frac{B_t}{A_t}. \tag{11}$$

Absent stickiness, with $\theta = 0$, the profit-maximizing price is the standard markup over current cost,

$$\tilde{p}_t = \frac{\epsilon}{\epsilon-1} z_t. \tag{12}$$

In period t, a fraction $(1-\theta)\theta^\tau$ of intermediate producers are still selling at a price that they reset last in period $t-\tau$. Thus the final good price in equation (2) is

$$p_t^{-(\epsilon-1)} = (1-\theta)\sum_\tau \theta^\tau \tilde{p}^{-(\epsilon-1)}. \tag{13}$$

The final good price p_t obeys the forward recursion,

$$p_t^{-(\epsilon-1)} = (1-\theta)\tilde{p}_t^{-(\epsilon-1)} + \theta p_{t-1}^{-(\epsilon-1)}. \tag{14}$$

Given the final-good price p, the embedded legacy price is:

$$\tilde{p}_t = \left(\frac{p_t^{-(\epsilon-1)} - \theta p_{t-1}^{-(\epsilon-1)}}{1-\theta}\right)^{-\frac{1}{\epsilon-1}}. \tag{15}$$

The technology for intermediate products is Cobb-Douglas with labor elasticity α. An intermediate product firm minimizes cost by setting the value of the marginal product of labor (total hours of work, h), valued at real marginal cost, to the real wage:

$$z\alpha h^{\alpha-1}k^{1-\alpha} = w. \tag{16}$$

Intermediate product producers share a common pool of installed capital, which rents for the real price,

$$b_t = q_{t-1}(r_t+\delta) - \Delta q_t. \tag{17}$$

Here q_t is the real price of a unit of installed capital—Tobin's q. Capital installation occurs up to the point where the marginal adjustment cost equals the difference between the price of installed capital q and the price of uninstalled capital, 1:

$$\kappa\frac{k_t - k_{t-1}}{k_{t-1}} = q_t - 1 \tag{18}$$

The parameter κ measures capital adjustment cost—if $\kappa = 0$, q is always 1 and there are no adjustment costs.

Unit cost is

$$z_t = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}w_t^\alpha b_t^{1-\alpha}. \tag{19}$$

The firm that last reset its price in period $t-\tau$ sets the level of labor input to

$$\alpha\frac{z_t\tilde{y}_{t,t-\tau}}{w_t}. \tag{20}$$

Here

$$\tilde{y}_{t,t-\tau} = y_t \left(\frac{\tilde{p}_{t-\tau}}{p_t} \right)^{-\epsilon} \tag{21}$$

is the level of output chosen by final good producers given the legacy price $\tilde{p}_{t-\tau}$. Aggregate labor input is

$$h_t = \sum_{\tau} (1-\theta)\theta^{\tau} \alpha \frac{z_t y_t}{w_t} \left(\frac{\tilde{p}_{t-\tau}}{p_t} \right)^{-\epsilon}. \tag{22}$$

Let

$$D_t = \sum_{\tau} (1-\theta)\theta^{\tau} \tilde{p}_{t-\tau}^{-\epsilon}. \tag{23}$$

Then

$$h_t = \alpha \frac{z_t y_t}{w_t} p_t^{\epsilon} D_t. \tag{24}$$

D_t obeys the forward recursion,

$$D_t = (1-\theta)\tilde{p}_t^{-\epsilon} + \theta D_{t-1}. \tag{25}$$

With flexible prices, $\theta = 0$,

$$h_t = \alpha \frac{z_t y_t}{w_t}, \tag{26}$$

the standard Cobb-Douglas labor demand.

Similarly, aggregate capital demand equals capital supply when

$$(1-\alpha)\frac{z_t y_t}{b_t} p_t^{\epsilon} D_t = k_{t-1}. \tag{27}$$

The economy's nominal anchor is a Taylor rule, stating the central bank's enforcement of a relation between the interest rate and rate of inflation:

$$r_t = \bar{r} + \phi \frac{p_t}{p_{t-1}} + \lambda \frac{y_t}{y^*}. \tag{28}$$

Here y^* is the stationary value of output in the model without sticky prices ($\theta = 0$).

The remainder of the model is a standard growth model with consumption-hours complementarity. At the beginning of a period, the stock of installed capital is k_{t-1}; people choose hours of work h_t. At the end of the period, output y_t becomes available and is allocated to government purchases g_t and consumption c_t. The rest of output plus the depreciated capital stock becomes the new capital stock at the beginning of period $t+1$, net of adjustment cost. The law of motion for capital is

$$k_t + \frac{\kappa}{2}\frac{(k_t - k_{t-1})^2}{k_{t-1}} = (1-\delta)k_{t-1} + y_t - c_t - g_t. \tag{29}$$

The interest rate r_{t+1} is the net marginal product of capital made available at the end of period t by a reduction in c_t, so consumers equate the marginal rate of substitution between c_t and c_{t+1} to $1 + r_{t+1}$.

Worker-consumers order their paths of hours and goods consumption according to the utility function

$$\sum_t \beta^t \left(\frac{c_t^{1-1/\sigma}}{1-1/\sigma} - \chi c_t^{1-1/\sigma} h_t^{1+1/\psi} - \gamma \frac{h_t^{1+1/\psi}}{1+1/\psi} \right), \tag{30}$$

where $\sigma < 1$ and $\chi \geq 0$ controls the complementarity of consumption and hours.

The first-order condition for the optimal mix of consumption and work is:

$$wc^{-1/\sigma} \left[1 - \chi(1-1/\sigma)h^{1+1/\psi}\right] = h^{1/\psi} \left[-\chi(1+1/\psi)c^{1-1/\sigma} - \gamma\right]. \tag{31}$$

The discounter is

$$m_{t,t+1} = \beta \frac{c_{t+1}^{-1/\sigma}}{c_t^{-1/\sigma}} \frac{1 - \chi(1-1/\sigma)h_{t+1}^{1+1/\psi}}{1 - \chi(1-1/\sigma)h_t^{1+1/\psi}}. \tag{32}$$

The Euler equation for consumption is

$$(1 + r_{t+1})m_{t,t+1} = 1. \tag{33}$$

Government purchases decline from an initial level g with a rate of persistence of ϕ:

$$g_t = g\phi^t. \tag{34}$$

2.1 Timing and boundary conditions

I use a timing convention suitable for Matlab: Period 1 describes the economy before a shock and provides initial conditions. The shock affects period 2 and later. The last economic decisions occur in period T. Capital at the end of period T is required to be at the economy's stationary level: $k_T = k^*$. For a reasonably large value of T, the result is very close to the infinite-horizon solution. I also set the auxiliary variables A and B to their stationary values in period T: $A_T = y^* p^{*(\epsilon-1)}$ and $B_T = z^* y^* p^{*\epsilon}$.

Variables requiring initial values in period 1 are capital k_1, the price level p_1, and the auxiliary variable D_1. I use the values $k_1 = k^*$, $p_1 = 1$, and $D_1 = 1$.

2.2 Summary of core model

Core variables: y, p, h, r, w, k, c, and q

$$\tilde{p}_t = \frac{\epsilon}{\epsilon - 1} \frac{B_t}{A_t} \tag{35}$$

$$h_t = \alpha \frac{z_t y_t}{w_t} p_t^{\epsilon} D_t \tag{36}$$

$$b_t = q_{t-1}(r_t + \delta) - \Delta q_t \tag{37}$$

$$(1-\alpha) \frac{z_t y_t}{b_t} p_t^{\epsilon} D_t = k_{t-1}. \tag{38}$$

$$\kappa \frac{k_t - k_{t-1}}{k_{t-1}} = q_t - 1 \tag{39}$$

$$r_t = \bar{r} + \phi \frac{p_t}{p_{t-1}} + \lambda \frac{y_t}{y^*} \tag{40}$$

$$w_t c_t^{-1/\sigma} \left[1 - \chi(1-1/\sigma) h_t^{1+1/\psi}\right] = h_t^{1/\psi} \left[-\chi(1+1/\psi) c_t^{1-1/\sigma} - \gamma\right] \tag{41}$$

$$(1 + r_{t+1}) m_{t,t+1} = 1 \tag{42}$$

$$k_t + \frac{\kappa}{2} \frac{(k_t - k_{t-1})^2}{k_{t-1}} = (1-\delta) k_{t-1} + y_t - c_t - g_t. \tag{43}$$

2.3 Summary of auxiliary model

Auxiliary variables: A, B, $\tilde{p}$, z, b, D, m, and g.

$$A_t = y_t p_t^{\epsilon-1} + \theta m_{t,t+1} A_{t+1} \tag{44}$$

$$B_t = z_t y_t p_t^{\epsilon} + \theta m_{t,t+1} B_{t+1} \tag{45}$$

$$\tilde{p}_t = \left(\frac{p_t^{-(\epsilon-1)} - \theta p_{t-1}^{-(\epsilon-1)}}{1-\theta} \right)^{-\frac{1}{\epsilon-1}} \tag{46}$$

$$z_t = \alpha^{-\alpha} (1-\alpha)^{-(1-\alpha)} w_t^{\alpha} b_t^{1-\alpha}. \tag{47}$$

$$D_t = (1-\theta) \tilde{p}_t^{-\epsilon} + \theta D_{t-1} \tag{48}$$

$$m_{t,t+1} = \beta \frac{c_{t+1}^{-1/\sigma}}{c_t^{-1/\sigma}} \frac{1 - \chi(1-1/\sigma)h_{t+1}^{1+1/\psi}}{1 - \chi(1-1/\sigma)h_t^{1+1/\psi}} \tag{49}$$

$$g_t = g\phi^t \tag{50}$$

2.4 Solution

The model has $9T - 11$ unknowns: Values of y, p, h, w, b, q, and c in periods 2 through T, r in periods 3 through T, and k in periods 2 through $T-1$. It has $9T - 11$ equations: the Euler equation spanning observation pairs $(2,3)$ through $(T-1,T)$, the equation for b for periods 3 through T, and the 6 other equations for periods 2 through T. I solve the equations for all periods simultaneously. The value b_2 is the shadow rental value of capital after the shock, given the level of k_2 that was determined prior to the shock.

3 Matlab Implementation

Function NKACCModel(x) takes a vector of stacked values of the core variables and calculates discrepancies for the core equations. It calculates auxiliary variables from the core variables as needed. The vector x is a solution to the model when NKACCModel delivers a vector of zeros to the designated precision.

Function StatModel(x) takes a vector $x = [c, h, K]$ and calculates discrepancies for the core equations that apply to the stationary version of the model.

Program NKACCMain uses Matlab library function fsolve and function StatModel to find the stationary values of all of the core variables. Then it solves for the vectors of values of the core variables for the dynamic solution using fsolve and NKACCModel. It can solve the model for a variety of parameter values in one run. It stores the results in Excel file NKACCResults.xlsx. Be sure this file is closed when running the Matlab program, as Matlab cannot write into an open Excel file.