

# U.S. Algebra Teaching and Learning Viewed Internationally

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# Outline

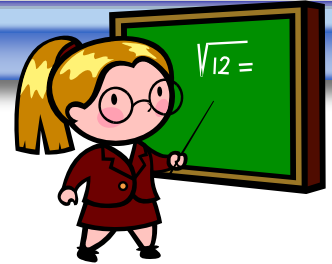


- Algebra in U.S. schools
- U.S. students' performance in TIMSS assessments
- Item characteristics
- High performance in algebra by U.S. students
- Low performance in algebra by U.S. students
- General observations

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# ***Algebra in U.S. schools***

# U.S. school algebra



- Separate high school courses
- Designed for the college bound
- Arose from college entrance requirements as algebra moved from the college to the high school
- Beginning algebra moving to Grade 8 and earlier
- Began as generalized arithmetic; has kept much of that character
- Slowly becoming more practical and functions based

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***U.S. students'  
performance in TIMSS  
assessments***

# Performance in TIMSS 1995

- U.S. 4th graders' performance in algebra, measured by number of items correct, was somewhat better than it was on all mathematics items taken together and better than the average for 4th graders in other countries
- U.S. 8th graders' performance in algebra was very slightly below its average on all items but about the same as that of 8th graders in other countries

# TIMSS 1995: Performance of U.S. fourth graders

	Number of items	U.S. average		26-country average	
Whole numbers	25	18	72%	17	68%
Fractions & proportionality	21	11	52%	10	48%
Measurement, estimation, & number sense	20	11	55%	11	55%
Data representation, analysis, & probability	12	9	75%	7	58%
Geometry	14	10	71%	9	64%
Patterns, relations, & functions	10	7	70%	6	60%
<b>Total</b>	<b>102</b>	<b>66</b>	<b>65%</b>	<b>60</b>	<b>59%</b>

# TIMSS 1995: Performance of U.S. eighth graders

	Number of items	U.S. average		41-country average	
Arithmetic	51	30	59%	30	59%
Geometry	23	11	48%	13	57%
Algebra	27	14	52%	14	52%
Probability & statistics	21	14	67%	13	62%
Measurement	18	7	39%	9	50%
Ratio & proportion	11	5	45%	5	45%
<b>Total</b>	<b>151</b>	<b>81</b>	<b>54%</b>	<b>84</b>	<b>56%</b>

# Performance in TIMSS 2003

- U.S. 4th graders' performance in algebra, measured by score points, was somewhat worse than it was on all mathematics items taken together but still better than the average for 4th graders in other countries
- U.S. 8th graders' performance in algebra was above its average on all items and above that of 8th graders in other countries

# TIMSS 2003: Performance of U.S. fourth graders

	Number of score points	U.S. average	26-country average
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Number	68	39	57%	34	50%
Patterns, equations, & relationships	24	13	54%	12	50%
Measurement	32	16	50%	17	53%
Geometry	25	16	64%	14	56%
Data	17	12	71%	10	59%
<b>Total</b>	<b>166</b>	<b>96</b>	<b>58%</b>	<b>87</b>	<b>52%</b>

# TIMSS 2003: Performance of U.S. eighth graders

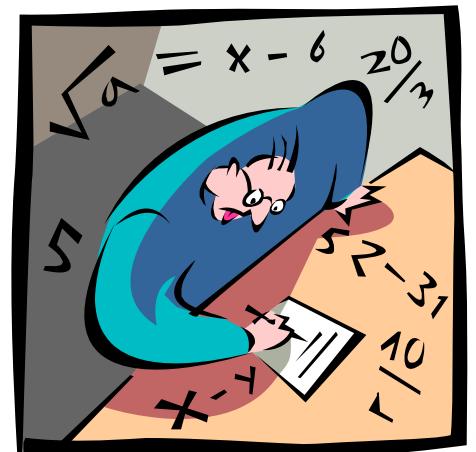
	Number of score points	U.S. average		45-country average	
Number	60	30	50%	26	43%
Algebra	53	29	55%	25	47%
Measurement	33	14	42%	13	39%
Geometry	33	14	42%	14	42%
Data	34	18	53%	13	38%
<b>Total</b>	<b>213</b>	<b>105</b>	<b>49%</b>	<b>91</b>	<b>43%</b>

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# ***Item characteristics***

# Item characteristics

- Content
- Representation
- Cognitive demand



# Content

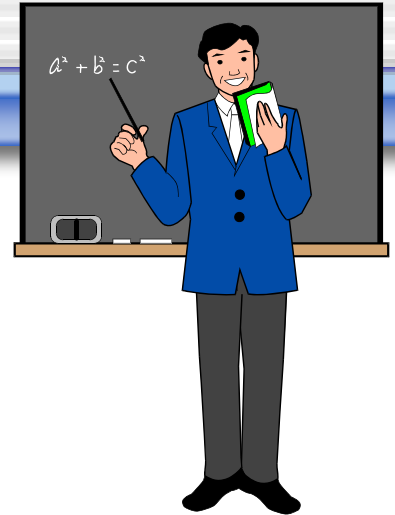


- Patterns
- Informal algebra
- Equations
- Inequalities
- Functions
- Algebraic reasoning
- Algebraic manipulation

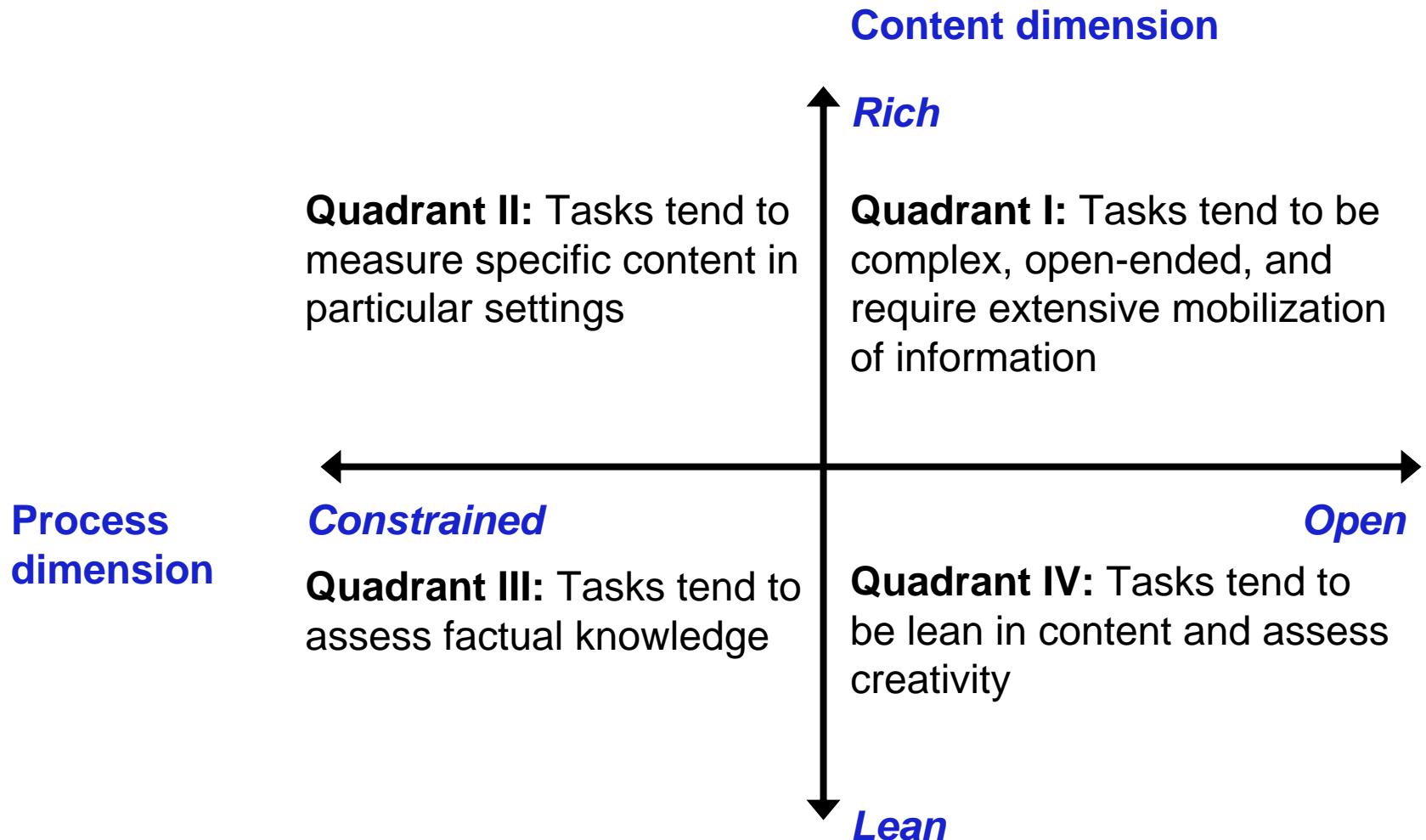
# Representation

- Numerical
- Verbal
- Graphical
- Symbolic
- Pictorial

- $N \rightarrow V$
- $V \rightarrow G$
- $S \rightarrow N$
- $P \rightarrow P$
- etc.



# Cognitive demand



# Seven systems

- Members of the Organisation for Economic Cooperation and Development (OECD)
- Participated in TIMSS 1995, 1999, 2003
- 1995 & 1999: Australia, **Canada**, **Czech Republic**, Hungary, the Netherlands, New Zealand, the United States
- 2003: Australia, **Ontario**, **Quebec**, Hungary, the Netherlands, New Zealand, the United States
- All but the US and New Zealand “A” countries (Valverde & Schmidt, 2000)

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***High performance in  
algebra by U.S. students***

# High performance

1995

Item L04 (Grade 4)

Item L13 (Grade 8)

Patterns,  $P \rightarrow P$ , Quadrant III (content lean-process constrained)

L4. These shapes are arranged in a pattern.

○△○○△△○○○△△△

Which set of shapes is arranged in the same pattern?

- A. ★□★□★★□□★★□□
- B. □★□□★□□□★□□□□
- C. ★□★★□□★★★□□□
- D. □□★★□★□□★★□★

Grade 4

Grade 8

U.S. percent correct:

79%

93%

International average:

72%

90%

# *High performance items*

- 4 in 1995, 5 in 1999, and 5 in 2003
- 4 at Grade 4, 8 at Grade 8, and 1 at Grade 12
- Tend to involve transition from arithmetic to algebraic thinking
  - ◆ Patterns
  - ◆ Informal algebra
- Numerical representations (solutions), rarely graphical and some verbal
- Mostly in Quadrant III (process constrained and content lean)

# Relative high performance

1999 Item P09 (Grade 8)

Algebraic manipulation,  $S \rightarrow S$ , Quadrant III

Which of these expressions is equivalent to  $n \times n \times n$  for all values of  $n$ ?

A.  $\frac{n}{3}$

B.  $n + 3$

C.  $3n$

D.  $n^3$

U.S. percent correct:

85%

Next highest percent correct (Czech Republic): 81%

# Relative high performance items

- 3 in 1995, 1 in 1999, 7 in 2003
- 6 at Grade 4, 5 at Grade 8
- Tend to involve transition from arithmetic thinking to algebraic thinking
  - ◆ Algebraic manipulation
  - ◆ Patterns
  - ◆ Informal algebra
- Do not involve pictorial or graphical representation
- Most located in Quadrant III (content lean and process constrained)

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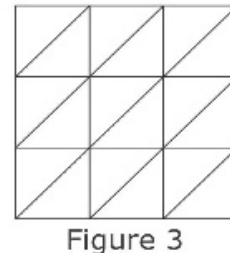
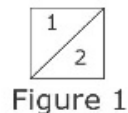
***Low performance in  
algebra by U.S. students***

# Low performance

2003 Item M04\_10c (Grade 8)

Functions,  $PN \rightarrow N$  or  $PN \rightarrow S$ , Quadrant I (content rich, open process)

The three figures below are divided into small congruent triangles.



- C. The sequence of figures is extended to the 50th figure. Explain a way to find the number of small triangles in the 50th figure that does not involve drawing it and counting the number of triangles.

U.S. percent correct:	22%
International average:	14%

# *Low performance items*

- 3 in 1995, 4 in 2003
- 1 in Grade 4, 5 in Grade 8, 1 in Grade 12
- Involve mostly equations and functions
- Involve mostly symbolic and verbal representations
- Located in Quadrants I and II (rich content, constrained or open process)

# Relative low performance

1999

Item V04a (Grade 8)

Patterns,  $P \rightarrow N$ , Quadrant III

The figures show four sets consisting of circles.



Figure 1



Figure 2

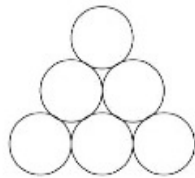


Figure 3

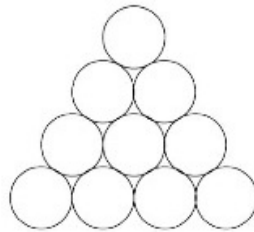


Figure 4

- a) Complete the table below. First, fill in how many circles make up Figure 4. Then, find the number of circles that would be needed for the 5th figure if the sequence of figures is extended.

Figure	Number of circles
1	1
2	3
3	6
4	
5	

U.S. percent correct (lowest):

73%

Hungary percent correct (2nd lowest): 77%

# Relative low performance items

- 8 in 1995, 5 in 1999, 4 in 2003
- All at eighth grade
- Most involve pattern generalization
- Most involve numerical or pictorial representations (no graphical)
- Most are in either Quadrant II (rich content, process constrained) or III (lean content, process constrained)
- Items are generally difficult for students in all selected countries

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# ***General observations***

# Observations about U.S. performance

- Fourth graders do relatively well in interpreting a rule, engaging in transitive reasoning, translating from words to symbols, and extending numerical patterns
- Twelfth graders can interpret a function graph, but their performance otherwise is relatively weak
- Eighth graders can understand exponential notation, interpret simple algebraic expressions, reason about sequences, and solve an equation for one variable in terms of another
- Their performance is relatively weak, however, in interpreting symbols in an equation, completing tables, finding sums of series, manipulating expressions, generalizing patterns, and solving word problems

# Final observations

- AIR study (Ginsburg et al., 2005): U.S. performance below the average for 12 countries at high and low levels of rigor for TIMSS items and at high and low levels of difficulty for TIMSS and PISA items
- Countries: Australia, Belgium, Hong Kong, Hungary, Italy, Japan, Latvia, Netherlands, New Zealand, Norway, the Russian Federation, and the United States
- Our results indicate, however, that although U.S. students' performance may be low in general, it is not uniformly low
- U.S. eighth graders do better in algebra than might be expected given their limited exposure to it

# Final observations



- AIR study also found that U.S. eighth-grade teachers far more likely (66% to 36% for other 11 countries) to say they relate mathematics to students' daily lives in most lessons
- TIMSS video studies show that U.S. teachers use few high complexity problems and spend 30% of lessons on review
- Algebra is of limited use if it is understood as generalized arithmetic only
- If students are to use algebra, they need to be proficient in functional thinking