Understanding of Symbols at the Transition from Arithmetic to Algebra: The Equal Sign and Letters as Variables

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Understanding Symbols

- Understanding symbols is key to success in algebra
- Symbols represent “core concepts” that are fundamental to algebra
  - Equal sign
  - Letters used as variables

    $3m + 7 = 25$

- Prior knowledge can be a stumbling block
  - Sets up expectations that may not apply
- Changes over time
  - Effects of experiences with symbols

Equal Sign

- What meanings do students ascribe to the equal sign? How do these change across grade levels? ...vary across task contexts?
  - Does prior knowledge influence students' interpretations?
- What is the relationship between meanings ascribed to the equal sign and performance on problems involving the equal sign?

Letters Used as Variables

- What meanings do middle school students ascribe to letters used as variables? How do these change across grade levels?
  - Does prior knowledge influence students' interpretations?
- What is the relationship between the meanings ascribed to variables and performance on problems that use variables?
- Does the form of the variable symbol affect students' interpretations?

Interpretations of the Equal Sign

- Past research has identified two main ways in which elementary and middle school students interpret the equal sign (e.g, Carpenter, Franke, & Levi, 2003; Kieran, 1981; McNeil & Alibali, 2005; Rittle-Johnson & Alibali, 1999)

- Operational
  – Equal sign is a signal to perform the given operations, put the answer
- Relational
  – Equal sign represents a relationship between quantities, indicates that two quantities are the same

Interpretation of the Equal Sign

The following questions are about this statement:

$$3 + 4 = 7$$

$$\qquad\quad\uparrow$$

(a) The arrow above points to a symbol. What is the name of the symbol?

(b) What does the symbol mean?

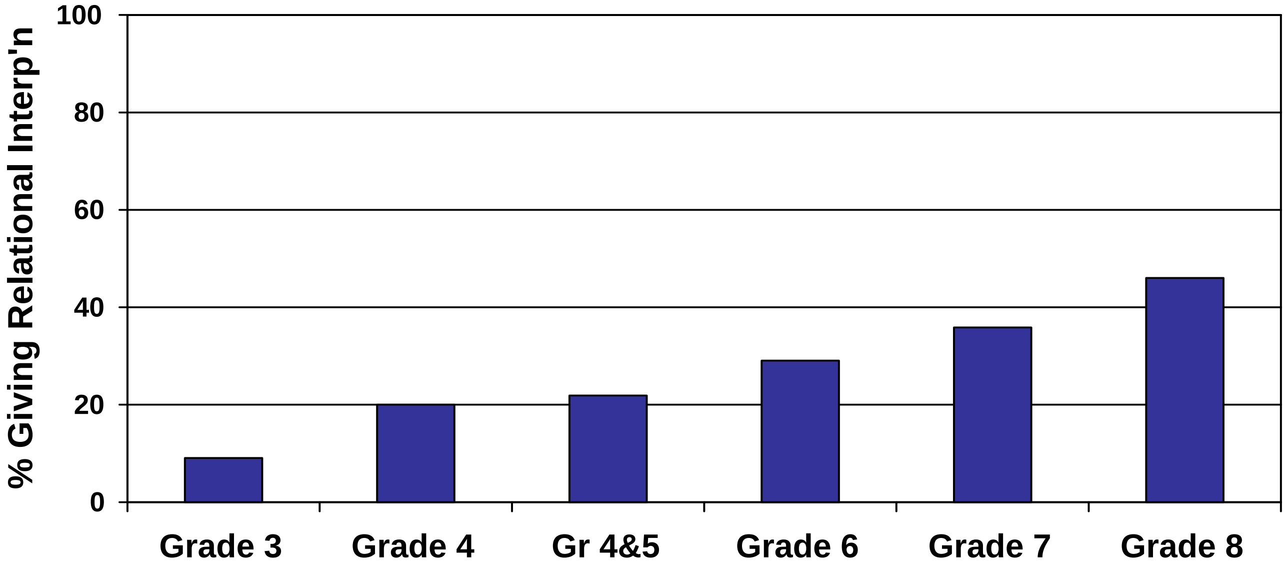
(c) Can the symbol mean anything else? If yes, please explain.

Sample Student Responses

Grades 5 - 7

- Operational
  - *The answer to the problem.*
  - *It's the symbol that goes before the answer. It tells you that the next number that goes there is the answer.*
  - *It means like what it's added up to.*
- Relational
  - *The same as.*
  - *Um sometimes we have these math problems, we have numbers and then other numbers and whichever number is greater you put a sign like an arrow, and whichever number is being used it would point to the littler number. Then if they're both the same you would put, I can't remember, 2 or 3 lines like this.*

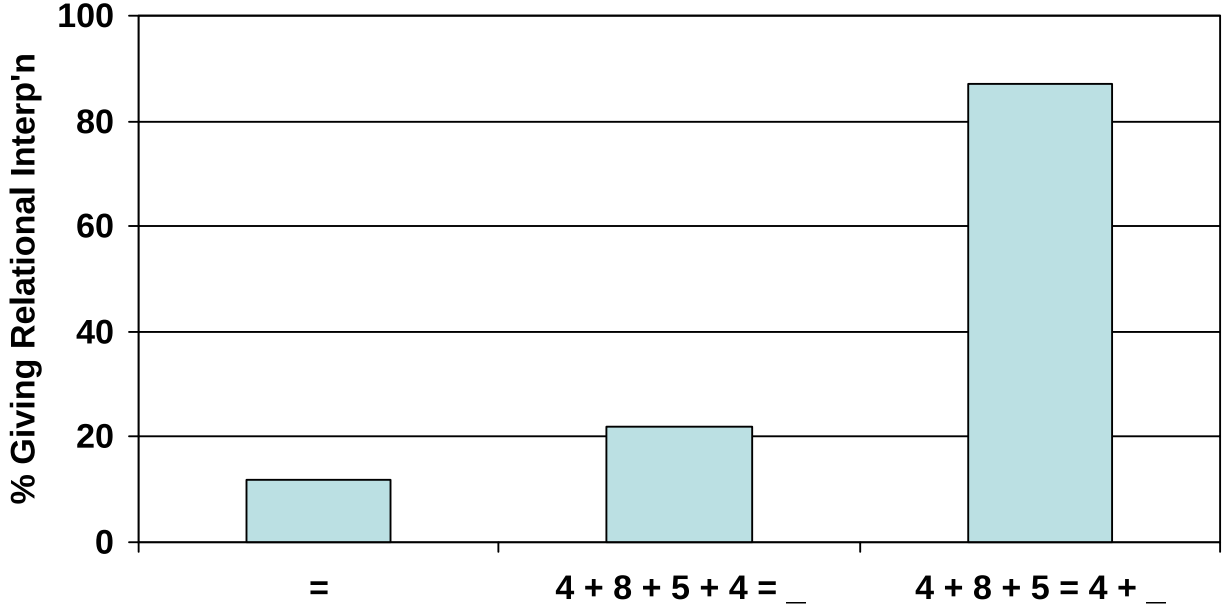
Improvement Across Grades



- Improvement across grade levels, BUT weak performance throughout the elementary and middle grades

Variability Across Task Contexts:

Grade 7 Students



• Students much more likely to offer relational interpretation in “operations on both sides” context

McNeil & Alibali, *J. Cog. & Dev.*, 2005

Does prior knowledge/experience influence students' interpretations?

- Elementary school mathematics involves extensive practice with arithmetic facts
  - Often in the form "operations = answer"

  - 3 + 8 = ?
  - 15 - 7 = __

QuickTime™ and a TIFF (LZW) decompressor are needed to see this picture.

- Most equations that elementary and middle school students encounter are in an "operations = answer" format (Seo & Ginsburg, 2001: McNeil, Grandau, et al., 2004)

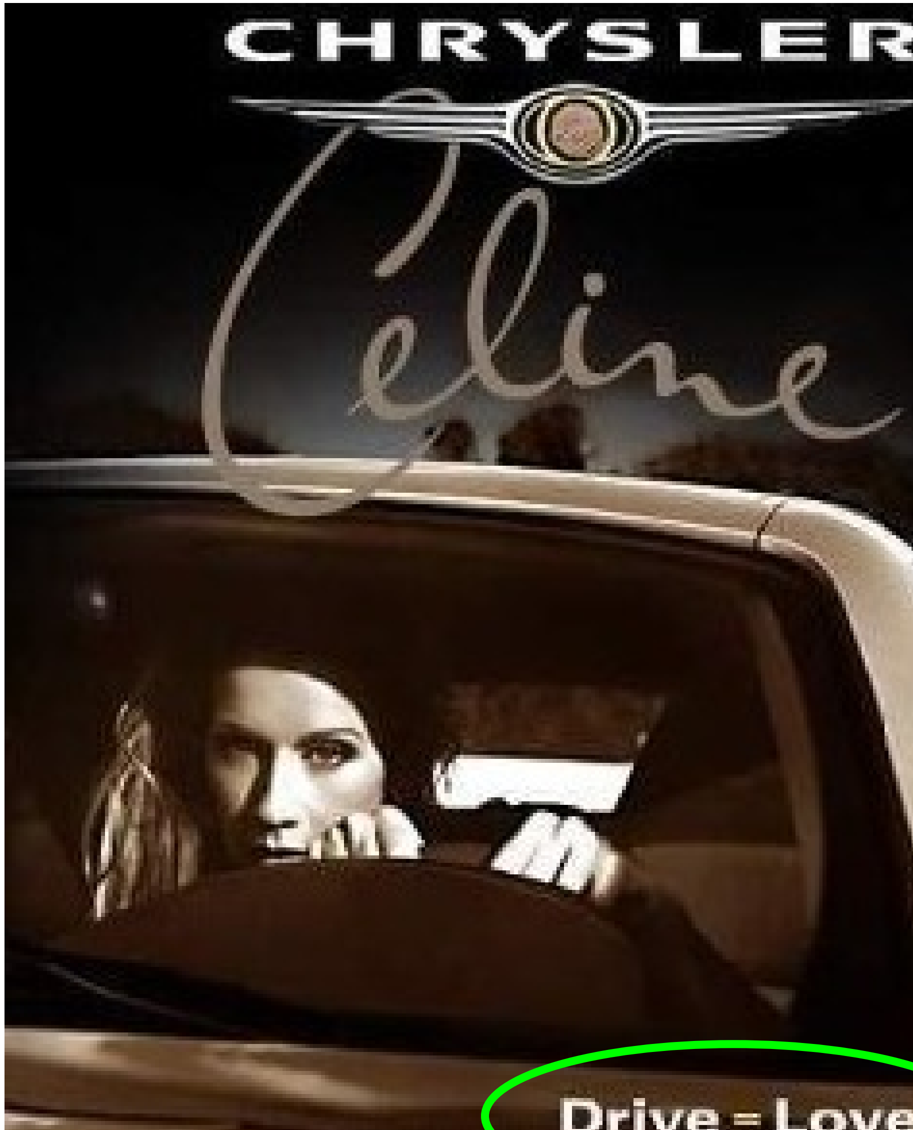
Does prior knowledge/experience influence students' interpretations?

- 4th- and 5th-grade students often misencode equations with operations on both sides as equation in the "operations = answer" format (McNeil & Alibali, 2004)

  E.g., asked to reconstruct $4 + 8 + 5 = 4 + \_\_$

  Many students write $4 + 8 + 5 + 4 = \_\_$

- Operational interpretation compatible with students' experiences, prior knowledge of how = is used in arithmetic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Drive = Love

Does understanding the equal sign matter?

- Do students who offer a relational interpretation perform better at tasks that involve the equal sign than students who offer an operational interpretation?

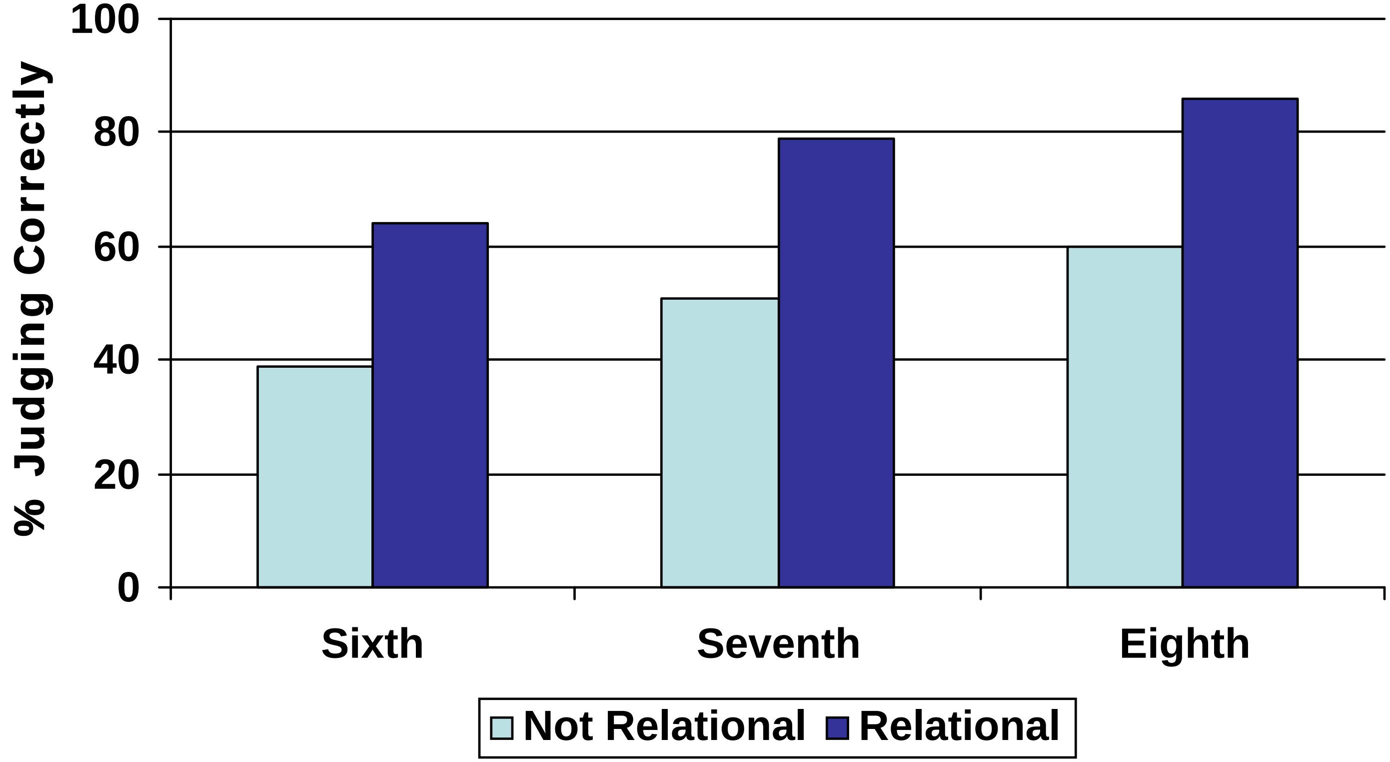
Equivalent Equations Problems

Is the number that goes in the $\square$ the same number in the following two equations? Explain your reasoning.

$2 \times \square + 15 = 31$

$2 \times \square + 15 - 9 = 31 - 9$

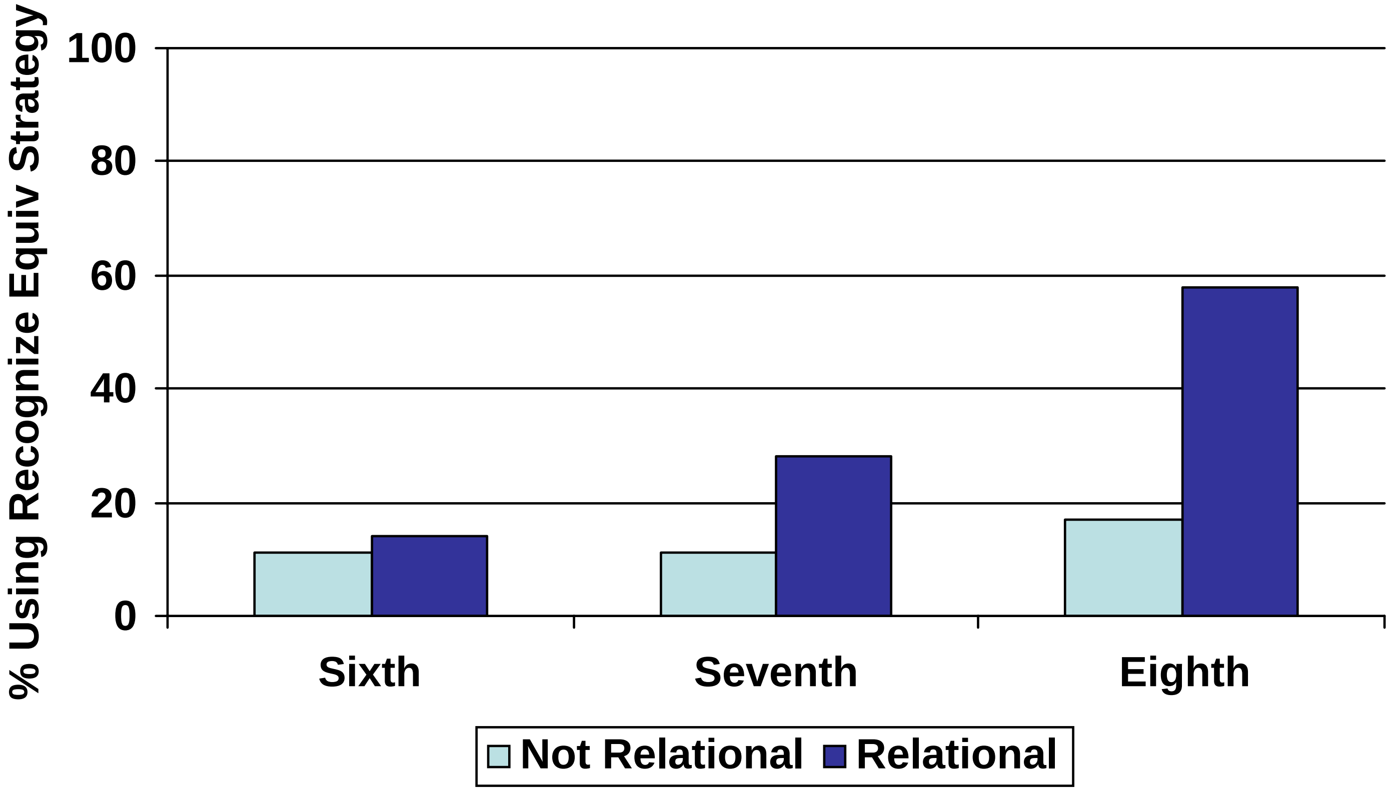
Equivalent Equations: Judgments



- More students with relational understanding judge that the solution for both equations is the same, *Wald* (1, N = 251) = 17.23, p < .01

Knuth, Alibali, et al., *Int'l Reviews in Math. Ed.*, 2005

Equivalent Equations: Strategy



• More students with relational understanding solve problem by recognizing equivalence, *Wald* (1, N = 251) = 16.10, p < .01

Knuth, Alibali, et al., *Int'l Reviews in Math. Ed.*, 2005

Linear Equations

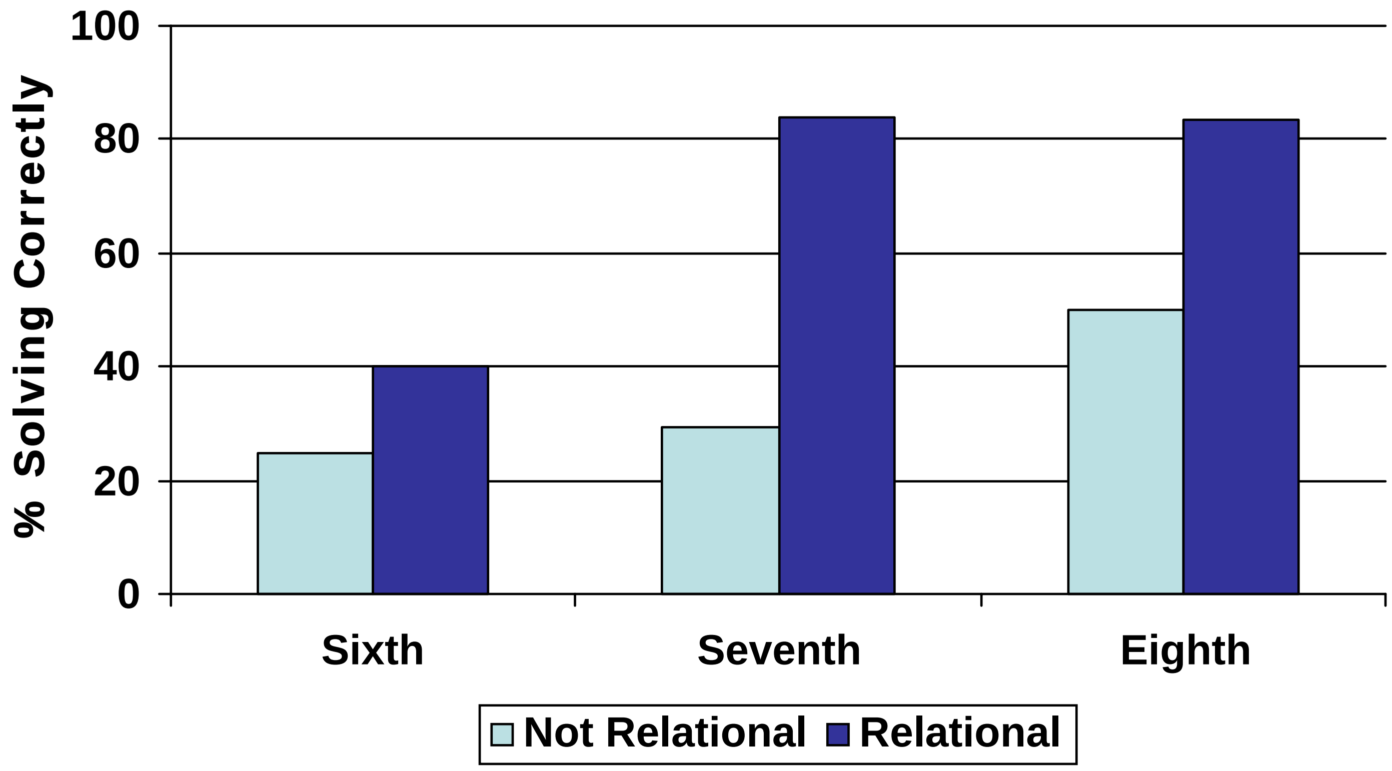
- What value of m will make the following number sentence true?

$4m + 10 = 70$

or

$3m + 7 = 25$

Linear Equations



- More students with relational understanding solve equations correctly, *Wald* (1, N = 177) = 22.64, $p < .01$
- Pattern holds even when controlling for mathematics ability, *Wald* (1, N = 65) = 3.85, $p = .05$

Knuth, Stephens, McNeil & Alibali, *J. Res. Math. Ed.*, in press

Does understanding the equal sign matter?

- Yes . . .
- Not simply that stronger students do better at both tasks

Letters Used as Variables

- What meanings do middle school students ascribe to letters used as variables? How do these change with grade level?
  - Does prior knowledge influence students' interpretations?
- What is the relationship between the meanings ascribed to variables and performance on problems that use variables?
- Does the form of the variable symbol affect students' interpretations?

Interpretations of Letters Used as Variables

- Past research has documented students' as well as adults' difficulties understanding letters used as variables (e.g., Clement, 1982; Küchemann, 1978; MacGregor & Stacy, 1997; Paige & Simon, 1966)
- One common misinterpretation is to treat letters as labels for objects -- an "abbreviation" interpretation
  - *The number of quarters a man has is seven times the number of dimes he has. The value of the dimes exceeds the value of the quarters by $2.50. How many of each coin does he have? (Paige & Simon, 1966)*

    d = dimes

    d = number of dimes

    d = value of dimes

Interpretation of a Letter Used as a Variable

The following question is about this expression:

2n + 3

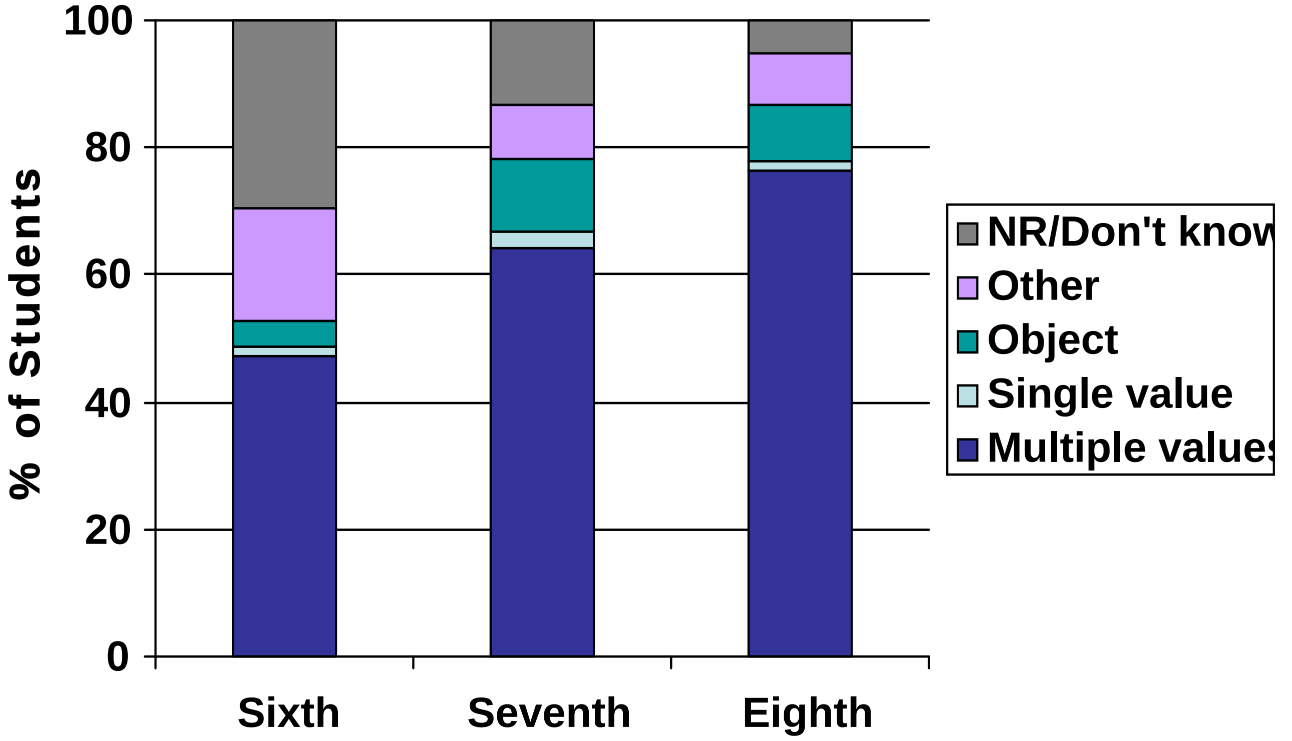
↑

The arrow above points to a symbol. What does the symbol stand for?

Coding Interpretations of Variables

- Multiple values
  - It can stand for any number.
- Single value
  - It stands for 4.
- Abbreviation
  - It means newspapers.
- Other

Variable Understanding as a Function of Grade Level



- Increase in multiple-values interpretations across the grades, *Wald* (1, *N* = 372) = 22.27, p < .01

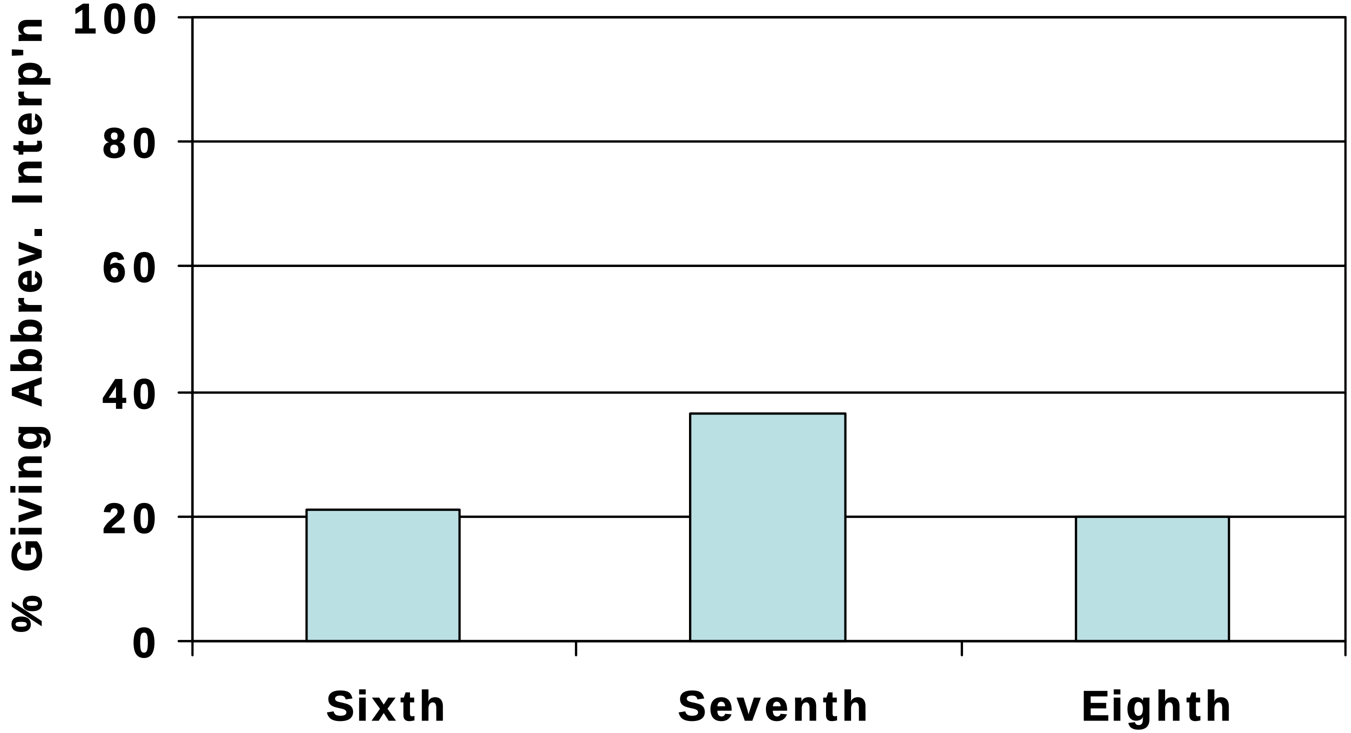
Knuth, Alibali, et al., *Int'l Reviews in Math. Ed.*, 2005

Does prior knowledge/experience influence students' interpretations?

- Students encounter letters as abbreviations in elementary school (e.g., initials, *m* for meter)
- Textbooks often use first letter as a variable symbol
- These experiences may promote the *abbreviation* interpretation
  - Rare in 2n + 3 item, but may be more common in items that involve objects

Cakes cost *c* dollars each and brownies cost *b* dollars each. Suppose I buy 4 cakes and 3 brownies. What does $4c + 3b$ stand for? (Küchemann, 1978)

Cakes & Brownies: Abbreviation Interpretation



- Abbreviation interpretation especially prevalent at 7th grade

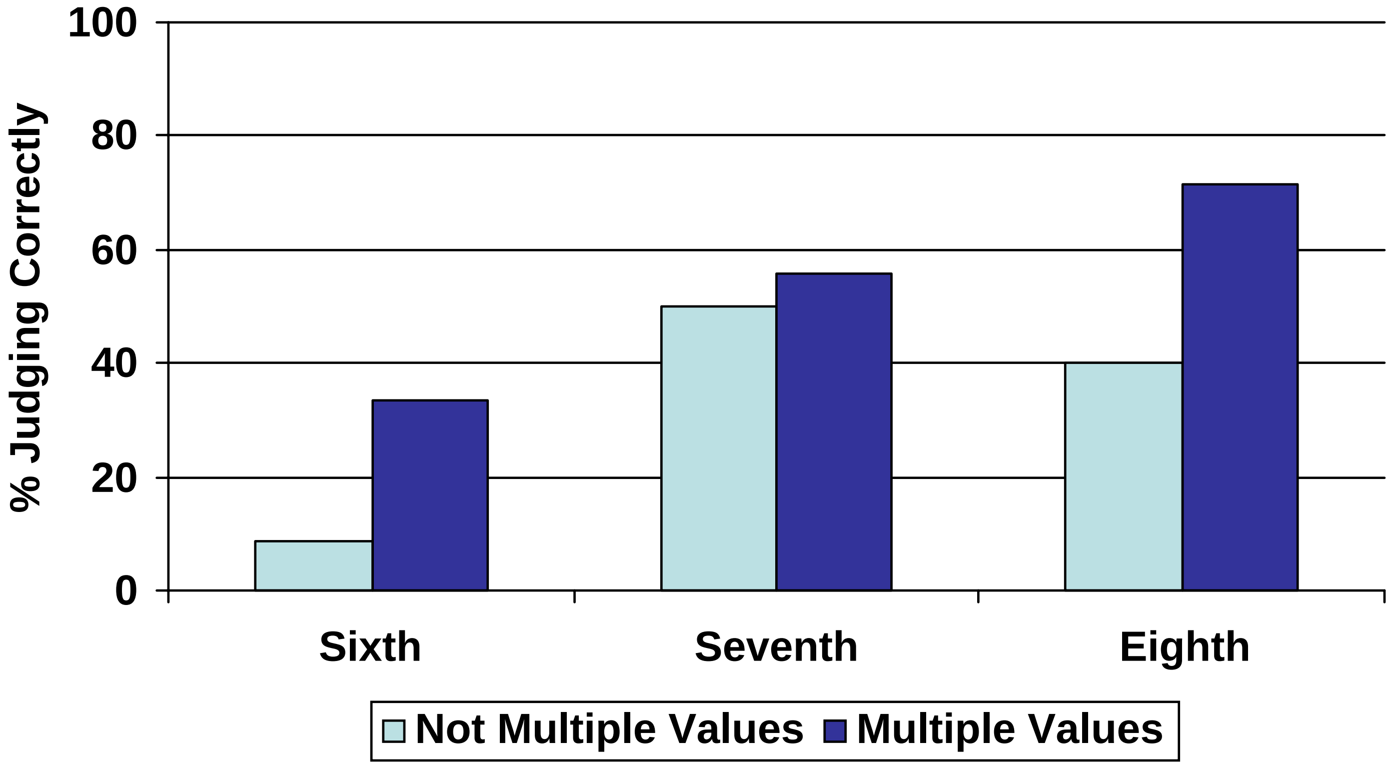
Does having a multiple-values interpretation matter?

- Do students who offer a multiple-values interpretation of letters used as variables perform better at tasks that involve letters used as variables than students who do not offer a multiple-values interpretation?

Using Variables: Which is Larger?

- Can you tell which is larger, $3n$ or $n + 6$? Please explain your answer.

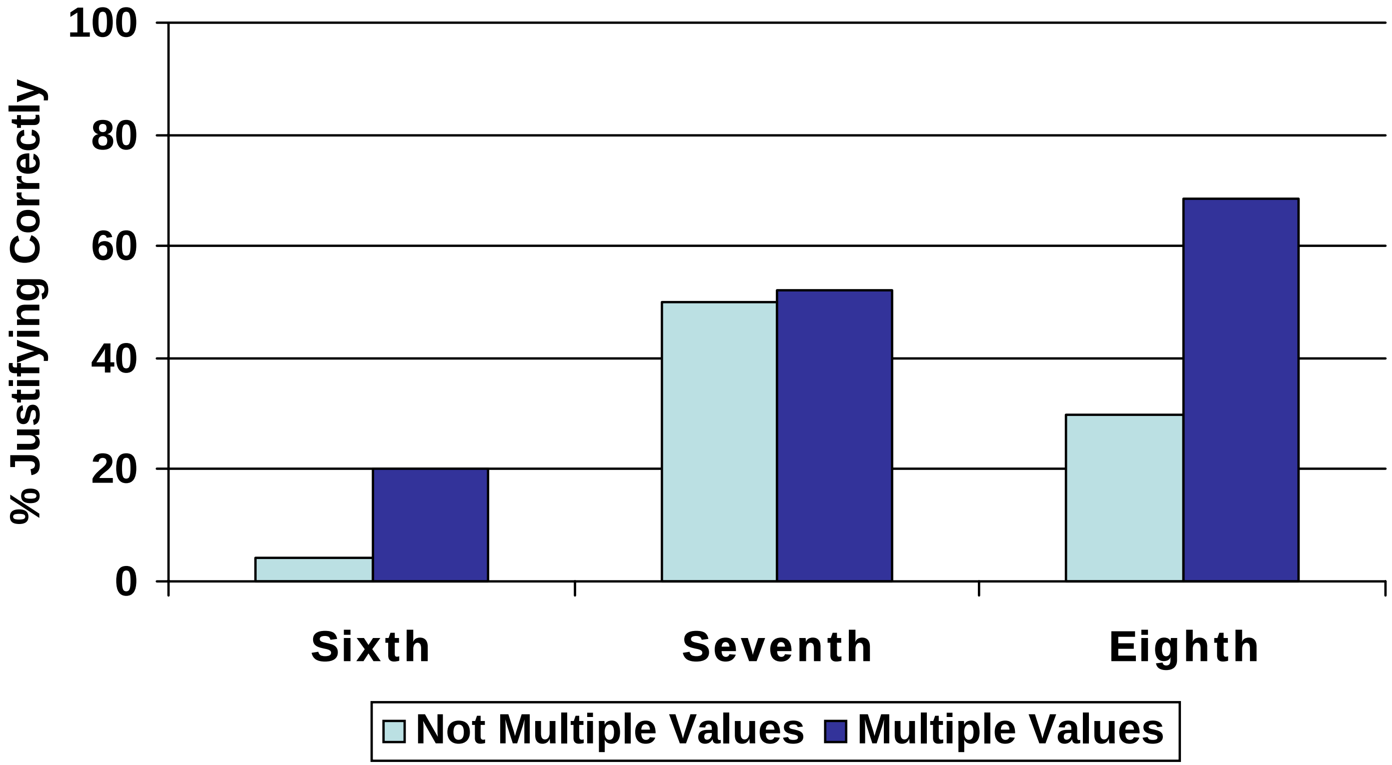
Which is Larger?: Judgments



- More students with multiple-values interpretations judge “can’t tell”, *Wald* (1, N = 122) = 4.9, $p < .03$

Knuth, Alibali, et al., *Int’l Reviews in Math. Ed.*, 2005

Which is Larger?: Justifications



- More students with multiple-values interpretations give correct justifications, *Wald* (1, *N* = 122) = 4.21, $p < .05$

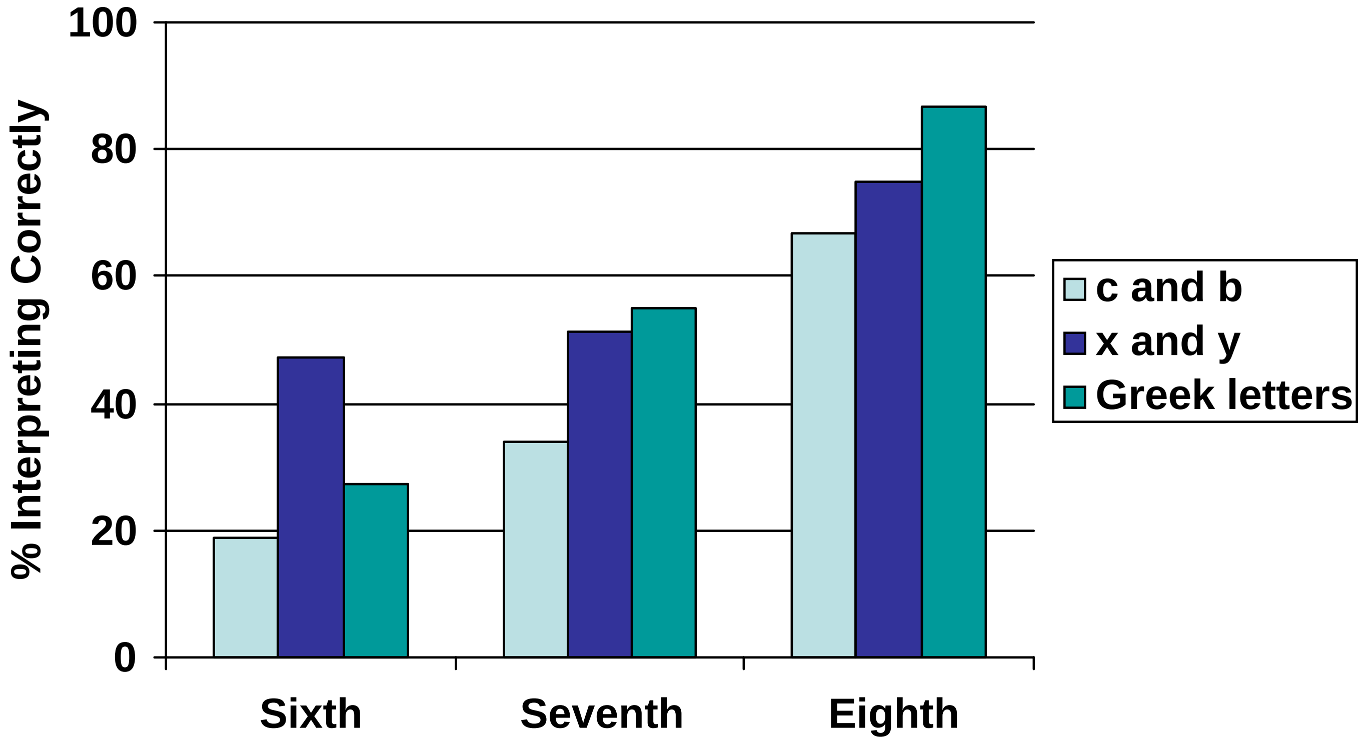
Knuth, Alibali, et al., *Int'l Reviews in Math. Ed.*, 2005

Interpretation of Letters Used as Variables

- Cakes cost c dollars each and brownies cost b dollars each. Suppose I buy 4 cakes and 3 brownies. What does $4c + 3b$ stand for?

- Do interpretations vary as a function of the specific symbols used?
  - b and c
  - x and y
  - Ψ and Φ

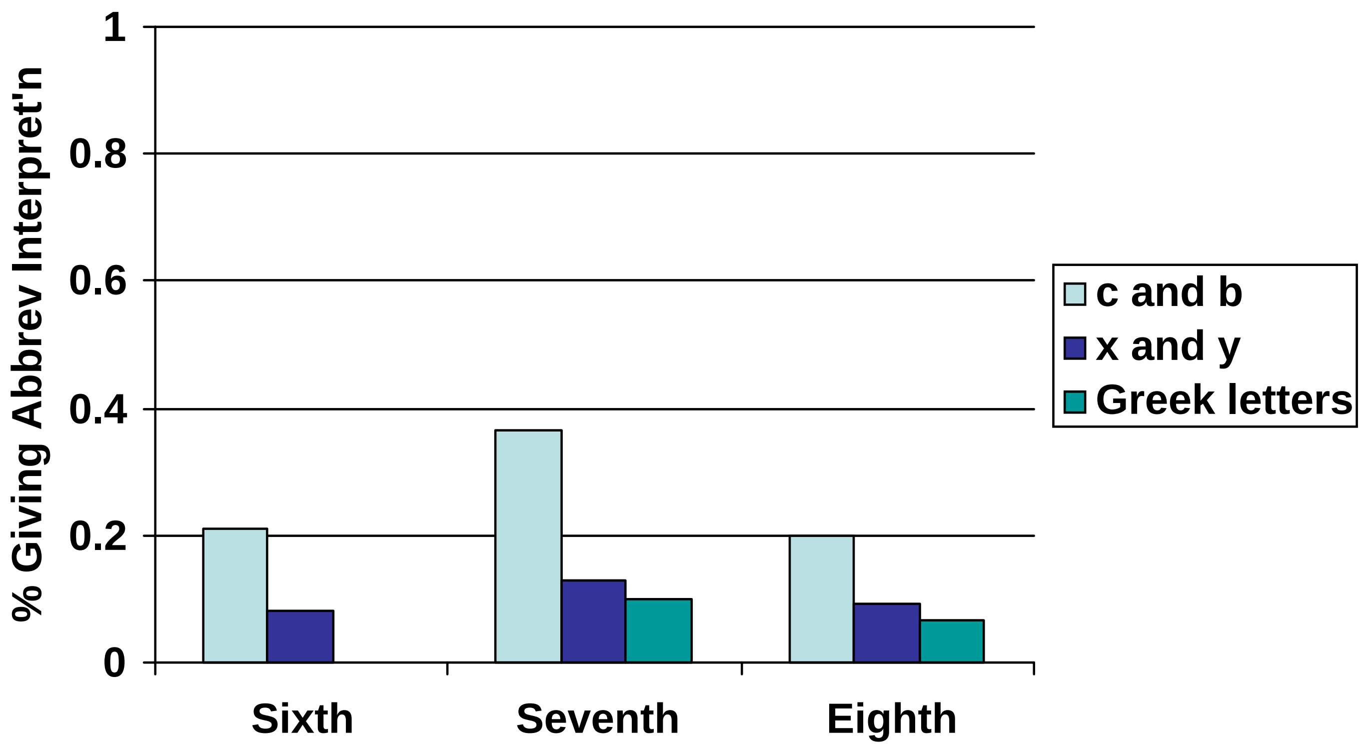
Correct Performance



- Students in *c and b* condition have weakest performance, *Wald* (1) = 9.34, p < .01

McNeil, Weinberg, Alibali & Knuth, *Soc. Res. Ch. Dev.*, 2005

Abbreviation Interpretation



• Students in *c and b* condition most likely to make abbreviation errors, *Wald* (1) = 19.34, p < .01

McNeil, Weinberg, Alibali & Knuth, *Soc. Res. Ch. Dev.*, 2005

Empirical Summary - Equal sign

- Many middle school students interpret the equal sign operationally rather than relationally
- Increase in relational definitions across the elementary and middle grades
- Students who have a relational understanding of the equal sign
  - More likely to judge that equivalent equations have the same solution
  - More likely to solve linear equations correctly

Empirical Summary - Variable

- Many middle-school students hold multiple-values interpretations of variables, but many do not
- Increase in multiple-values interpretations across the middle grades
- Students who have multiple-values interpretations
  – More likely to judge that one “can’t tell” whether 3n or n+6 is larger
  – More likely to provide a correct justification
- Students often interpret literal symbols as abbreviations
  – “First letter” symbols especially prone to this misinterpretation

Learning Meanings of Symbols

- Students derive expectations about meanings of symbols from prior experiences
  - Often, experiences not intended to be algebraic
- Students may also infer meanings based on implicit learning about contexts in which symbols occur
  - E.g., "operations = answer" contexts for the equal sign
- Students may also have opportunities for explicit learning
  - However, little is known about what types of instruction are most effective
    - Bridging from prior knowledge
    - Contrasting cases

Learning Meanings of Symbols at the Transition to Algebra

- Development of understanding of symbols depends crucially on opportunities for learning about symbols
- Patterns of performance suggest that, in some respects, prior knowledge may be a stumbling block

- Educators should be thoughtful about providing opportunities for learning, both explicit and implicit

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